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Systems of reaction–convection–diffusion equations invariant under Galilean algebras

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ABSTRACT

Systems, which are invariant under Galilean algebras (with and without mass operator), and their main extensions by operators of scale and projective transformations are selected from a class of systems of reaction–convection–diffusion equations for two-dimensional vector field with one space variable.

1. Introduction

A research of many physical processes requires constructing mathematical models in a form of differential equations or systems of differential equations, which are based on laws or properties inherent in this process. Mathematical models, which are used to describe physical processes, usually contain arbitrary functions. So, they form whole classes of differential equations or their systems. There is a question: how to choose from a set of logically acceptable models the ones, which are more suitable for describing a specific physical process than the others?

A long time ago, Jacobi has formulated conservation laws of classical mechanics on the basis of principles of symmetry [22]. Later, Klein has analyzed equations of general theory of relativity from the same point of view and emphasized an importance of the research of the group-theoretic nature of conservation laws for differential equations [25]. In 1918, combining formal methods of calculus of variations and the theory of Lie groups, E. Noether has formulated her theorem, according to which, the invariance of some properties of a system corresponds to a certain conservation law [39]. The presence of symmetry in a system determines the existence of a physical quantity, which doesn't change, concerning this system. For differential equations,

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symmetry can be regarded as a principle by which only those models, which have a wide symmetry, can be selected from a variety of acceptable models (equations, relations, etc.). In fact, all basic equations of mathematical physics (Newton's, Laplace's, d'Alembert's, Schrödinger's, Liouville's, Dirac's and Maxwell's equations, etc.) are invariant under rather wide groups of transformations [13].

Thus, a selection from a particular class such systems, which have wide symmetry properties and, because of this, are more suitable for describing certain physical processes, is an actual problem.

We consider a system of reaction–convection–diffusion equations

$$U_0 = \partial_1 \big[F(U)U_1\big] + G(U)U_1 + H(U) \tag{1}$$

where $U = \binom{u^1}{u^2}$, $u^a = u^a(x)$, $x = (x_0, x_1)$, $F(U) = (f^{ab}(u))$, $G(U) = (g^{ab}(u))$ – arbitrary functional matrices of the second order, $H(U) = (h^a(u))$ – an arbitrary functional matrix of 2×1 order, $a \in \{1, 2\}$, subscripts mean partial derivatives. System (1) with certain nonlinearities $F(U)$, $G(U)$ and $H(U)$ is applied for modeling of various processes of physics, chemistry, biology and ecology. Thus, modifications of system (1) are used for description of transfer processes in a body, such as modeling of an oxygen transportation in a circulatory system or modeling of a growth of thrombus in a wall flux. One of possible applications of system (1) in ecology is study of a spread of pollutants in the water. In biology, a system of reaction–diffusion–advection equations describes a model of a predator–prey community, a bacterial motion in colonies, influenced by various factors, etc. Hydrodynamic instability at the interface between two immiscible liquids is observed in such industries as oil refining, combustion processes, separation of ores, etc. A mathematical model of this phenomenon also involves a system of equations of reaction–convection–diffusion. A transition to a complex variable in system (1) allows obtaining models, which describe a motion of a quantum particle (Schrödinger equation), a state of a superconductor in an external magnetic field (Ginzburg–Landau equation) and magnetohydrodynamic waves in plasma.

From the mathematical point of view, Eq. (1) describes not only one system, but a whole class of them. Because of this it is very important to select from this class exactly those systems, which can describe specific natural phenomena. A powerful method of choosing such systems is a method of symmetry. This method allows to select those systems, which have wide symmetry properties, satisfy a principle of relativity and can be used for description of physical processes.

It should be noted that symmetry properties of system (1) for different specific sets of matrices $F(U)$, $G(U)$ and $H(U)$ have been studied by many authors. For instance, in paper [10] the Lie symmetry of a reaction–convection–diffusion equation has been researched. When $G = 0$, we obtain from system (1) a system of reaction–diffusion equations, whose symmetry properties have also been a subject of many papers. Thus, in [4–7,33–38] significant results have been obtained for different kinds of constant matrices of diffusion.

Introducing certain restrictions for matrices F and $G = 0$ we obtain a system of equations of chemotaxis, whose symmetry properties have been studied in publication [47]. Besides this, a convection–diffusion system can be obtained from system (1) (with $H = 0$). Works [8] and [9] are devoted to researching of Lie and Q-conditional symmetry of a two-dimensional system of this kind; in the case of a unit matrix F, the invariance of a three-dimensional system of convection–diffusion equations in regard to a generalized Galilean algebra has been studied in [46], in the case of arbitrary constant matrix F similar studies have been made for a two-dimensional system in [48].

2. Problem statement and definition

It is well known (e.g., see [12,27]), that a linear heat equation

$$u_0 = u_{11}$$

is invariant under the following algebra of operators

$$\partial_0, \quad \partial_1, \quad G = x_0\partial_1 + x_1 Q_0, \quad Q_0, \tag{2}$$

$$D = 2x_0\partial_0 + x_1\partial_1 + Q_2, \tag{3}$$

$$\Pi = x_0^2\partial_0 + x_0x_1\partial_1 + x_0Q_2 + \frac{x_1^2}{2}Q_0 \tag{4}$$

where $Q_0 = -Q_2 = -\frac{1}{2}u\partial_u$.

One more equation, which is invariant under a similar algebra, is Burgers' equation

$$u_0 + uu_1 = u_{11}$$

whose maximal algebra of invariance (e.g., see [24,45]) is the algebra with basic generators

$$\partial_0, \quad \partial_1, \quad G = x_0\partial_1 + Q_1, \tag{5}$$

$$D = 2x_0\partial_0 + x_1\partial_1 + Q_2, \tag{6}$$

$$\Pi = x_0^2\partial_0 + x_0x_1\partial_1 + x_0Q_2 + x_1Q_1 \tag{7}$$

where $Q_1 = \partial_u$, $Q_2 = -u\partial_u$.

Since operators G, given by formula (2) or (5), generate the Galilean transformations

$$x_0' = x_0, \qquad x_1' = x_1 + \theta x_0$$

of space (x_0, x_1) (e.g., see [17]), they are called the Galilean operators. Algebra (2) is called the Galilean algebra with mass operator Q_0, and algebra (5) is known as the Galilean algebra without mass operator. Algebras (2)–(3) or (5)–(6) are called the extended Galilean algebras, and algebras of operators (2)–(4) or (5)–(7) are the generalized Galilean algebras.

In this paper we describe all the systems of class (1), which are invariant under the Galilean algebras (with and without mass operator), supplemented by operators of large-scale and projective transformations.

Operators of the algebra (5)–(7) satisfy the following commutation relations:

$$[\partial_1, G] = 0, \tag{8}$$

$$[\partial_0, \partial_1] = 0, \qquad [\partial_0, G] = \partial_1, \tag{9}$$

$$[\partial_0, D] = 2\partial_0, \qquad [\partial_1, D] = \partial_1, \qquad [G, D] = -G, \tag{10}$$

$$[\partial_0, \Pi] = D, \qquad [\partial_1, \Pi] = G, \qquad [G, \Pi] = 0, \qquad [D, \Pi] = 2\Pi. \tag{11}$$

In this regard, a three-dimensional algebra $\langle X_1, X_2, X_3\rangle$, whose operators satisfy commutation relations (8)–(9), is called the Galilean algebra without mass operator and marked as $AG(1,1)$; a four-dimensional algebra $\langle X_1, X_2, X_3, X_4\rangle$, whose operators satisfy commutation relations (8)–(10), is called the extended Galilean algebra with mass operator and marked as $AG_1(1,1)$; and a five-dimensional algebra $\langle X_1, X_2, X_3, X_4, X_5\rangle$, whose operators satisfy commutation relations (8)–(11), is called the generalized Galilean algebra without mass operator and marked as $AG_2(1,1)$.

Operators of the algebra (2)–(4) satisfy commutation relations (9)–(11) and the following commutating relations:

$$[\partial_1, G] = Q_0, \qquad [\partial_0, Q_0] = 0, \qquad [\partial_1, Q_0] = 0, \qquad [G, Q_0] = 0, \tag{12}$$

$$[D, Q_0] = 0, \tag{13}$$

$$[\Pi, Q_0] = 0. \tag{14}$$

In this regard, a four-dimensional algebra $\langle X_1, X_2, X_3, X_4\rangle$, whose operators satisfy commutation relations (9) and (12), is called the Galilean algebra with mass operator and marked $AG^M(1,1)$; a five-dimensional algebra $\langle X_1, X_2, X_3, X_4, X_5\rangle$, whose operators satisfy commutation relations (9), (10) and (12), (13), is called the extended Galilean algebra with mass operator and marked as $AG_1^M(1,1)$; and a six-dimensional algebra $\langle X_1, X_2, X_3, X_4, X_5, X_6\rangle$, whose operators satisfy commutation relations (9)–(11) and (12)–(14), is called the generalized Galilean algebra with mass operator and marked as $AG_2^M(1,1)$. The numbers in the brackets mean that the space of independent variables of the system of differential equations consists of two variables: one temporal x_0 and one spatial x_1.

3. The system of determining equations. Converting the equivalence of system (1)

Using the infinitesimal Lie method, we find a system of determining equations for nonlinearities $F(U)$, $G(U)$, $H(U)$ and coordinates of the infinitesimal operator of the group of symmetry of system (1).

The following statement is true.

Lemma 1. *The system (1) is invariant under the infinitesimal operator*

$$X = \xi^\mu(x,u)\partial_\mu + \eta^a(x,u)\partial_{u^a} \tag{15}$$

if and only if functions ξ^μ, η^a, F, G and H satisfy the following system of determining equations

$$\xi^0_1 = \xi^\mu_{u^a} = 0, \tag{16}$$

$$f^{db}\eta^a_{u^c u^d} + f^{dc}\eta^a_{u^b u^d} = 0, \tag{17}$$

$$\eta^c f^{ab}_{u^c} + \left(\xi^0_0 - 2\xi^1_1\right) f^{ab} + \eta^c_{u^b} f^{ac} - \eta^a_{u^c} f^{cb} = 0, \tag{18}$$

$$\eta^c g^{ab}_{u^c} + \left(\xi^0_0 - \xi^1_1\right) g^{ab} + \eta^c_{u^b} g^{ac} - \eta^a_{u^c} g^{cb} + \eta^c_1\left(f^{ab}_{u^c} + f^{ac}_{u^b}\right) + 2\eta^c_{1u^b} f^{ac} - \xi^1_{11} f^{ab} + \delta_{ab}\xi^1_0 = 0, \tag{19}$$

$$\eta^c h^a_{u^c} + \xi^0_0 h^a - \eta^a_{u^c} h^c + \eta^b_{11} f^{ab} + \eta^b_1 g^{ab} - \eta^a_0 = 0 \tag{20}$$

where f^{ab}, g^{ab} and h^a are components of functional matrices, respectively $F(U)$, $G(U)$, $H(U)$, $a,b,c,d \in \{1,2\}$, $\mu \in \{0,1\}$, δ_{ab} is the Kronecker symbol, and subscripts mean partial derivatives.

Lemma 1 is proved by S. Lie's standard method (e.g., see [27,40,41]).

Definition 1. We call an algebra, for which the system is invariant with arbitrary nonlinearities F, G, H, the basic algebra of invariance of Eq. (1).

As a consequence of Lemma 1, the following statement is right.

Lemma 2. *The basic algebra of invariance of system (1) is the algebra*

$$A^{bas} = \langle \partial_0, \partial_1 \rangle. \tag{21}$$

Proof. If we split the system of determining equations (16)–(20) accordingly to arbitrary elements f^{ab}, g^{ab}, h^a and their derivatives, we obtain

$$\xi^0 = c_0, \qquad \xi^1 = c_1, \qquad \eta^a = 0 \tag{22}$$

where c_0, c_1 are arbitrary constants. Operator (15) with coordinates (22) generates algebra (21).

Lemma 2 is proved. □

An important role in the group classification of a class of differential equations belongs to the equivalent transformation of that class.

Definition 2. Equivalent transformations are transformations of dependent and independent variables, which transform an arbitrary equation (or a system of equations) from any class of differential equations to a differential equation (a system of equations) of the same class.

Knowing equivalent transformations we can divide a class of differential equations into nonequivalent subclasses, find a canonical representative in every subclass, study its symmetry properties and, by means of these transformations, extend the obtained results to all equations of that subclass.

The following statement is true.

Lemma 3. *The group of continuous transformations of equivalence for system (1) is the group*

$$x_0' = a_0 x_0 + b_0, \qquad x_1' = a_1 x_1 + g x_0 + b_1, \tag{23}$$

$$u^{a\,\prime} = \alpha_{ab} u^b + \beta_a \tag{24}$$

where a_μ, b_μ, α_{ab}, β_a, g are arbitrary constants.

The proof of Lemma 3 is carried out by the method described in the paper [1].

All further considerations are made to within equivalence transformations (23) and (24).

4. Representations of the Galilean algebras

4.1. Representations of the Galilean algebra $AG(1,1)$ and its extensions

We set the view of operators of Galilean algebras without mass operator, under which system (1) can be invariant. We clarify the general appearance of infinitesimal operator (15) for system (1), satisfying Eqs. (16), (17).

From system of equations (16) it implies that $\xi^0 = A(x_0)$, $\xi^1 = B(x_0, x_1)$, where $A(x_0)$, $B(x_0, x_1)$ are arbitrary smooth functions of their arguments. We consider system of equations (17) as a linear algebraic system with respect to unknown functions $\eta^a_{u^b u^c}$. The determinant of this system is $\Delta = \langle 1\rangle \cdot \langle 2\rangle$ where the expressions

$$\langle 1\rangle = f^{11} + f^{22}, \qquad \langle 2\rangle = \begin{vmatrix} f^{11} & f^{12} \\ f^{21} & f^{22} \end{vmatrix}$$

are the traces of matrix $F = (f^{ab})$ of the 1st and 2nd orders respectively.

Since at $\langle 2\rangle = 0$ system (1) describes processes associated with the behavior of a two-phase fluid, and at $\langle 1\rangle = 0$ system (1) is an equation of the Schrödinger type for a complex function $\psi = u^1 + iu^2$, we assume that

$$\Delta \neq 0. \tag{25}$$

Linear system (17) is homogeneous with respect to variables $\eta^a_{u^b u^c}$ with condition (25) and has only the trivial solution

$$\eta^a_{u^b u^c} = 0. \tag{26}$$

The general solution of system (26) is $\eta^a = \alpha^{ab}(x_0, x_1)u^b + \beta^a(x_0, x_1)$ where $\alpha^{ab}(x_0, x_1)$, $\beta^a(x_0, x_1)$ are arbitrary smooth functions of their arguments.

Thus we have defined the following statement.

Theorem 1. *If system (1) with condition (25) is invariant under the operator (15), then this operator has the form*

$$X = A(x_0)\partial_0 + B(x_0, x_1)\partial_1 + \left[\alpha^{ab}(x_0, x_1)u^b + \beta^a(x_0, x_1)\right]\partial_{u^a} \tag{27}$$

where A, B, α^{ab}, β^a are arbitrary smooth functions of their arguments.

Since system (1) with arbitrary nonlinearities F, G, H is invariant under algebra (21), we take operators $X_1 = \partial_0$, $X_2 = \partial_1$ as two operators of the Galilean algebra and, as Theorem 1 implies, we search for the third operator of this algebra in the form (27), where $A(x_0)$, $B(x_0, x_1)$, $\alpha^{ab}(x_0, x_1)$ and $\beta^a(x_0, x_1)$ are unknown functions.

Using commutation relations (8)–(9), we get

$$\dot{A} = B_1 = 0, \qquad B_0 = 1, \qquad \alpha_0^{ab} = \alpha_1^{ab} = \beta_0^a = \beta_1^a = 0. \tag{28}$$

Solving Eqs. (28), we obtain that operator X_3 is

$$X_3 = c_0\partial_0 + (x_0 + c_1)\partial_1 + \left[\alpha_{1ab}u^b + \beta_{1a}\right]\partial_{u^a}$$

where c_0, c_1, α_{1ab}, β_{1a} are constants of integration. So, considering $G = X_3 - c_0\partial_0 - c_1\partial_1$, we get the realization of algebra (8)–(9) for system (1):

$$AG(1,1) = \langle \partial_0,\ \partial_1,\ G = x_0\partial_1 + Q_1 \rangle, \tag{29}$$

where $Q_1 = (\alpha_{1ab}u^b + \beta_{1a})\partial_{u^a}$.

In paper [42], to within all possible local transformations, non-equivalent realizations of algebras of 1–4 orders were determined. Among the algebras presented in this paper was algebra (29), but provided that $Q_1 = \partial_{u^1}$. Since system (1) does not allow all the possible equivalence transformations, but only transformations in the form (23), (24), the class of operators for it is wider.

Similarly, using commutation relations (8)–(11), we conclude that the basic generators of the extended and generalized Galilean algebras without mass operator, under which system (1) can be invariant, are

$$AG_1(1,1) = \langle \partial_0,\ \partial_1,\ G = x_0\partial_1 + Q_1,\ D = 2x_0\partial_0 + x_1\partial_1 + Q_2 \rangle, \tag{30}$$

$$\begin{aligned} AG_2(1,1) = \big\langle \partial_0,\ \partial_1,\ G = x_0\partial_1 + Q_1,\ D = 2x_0\partial_0 + x_1\partial_1 + Q_2, \\ \Pi = x_0^2\partial_0 + x_0x_1\partial_1 + x_1Q_1 + x_0Q_2 + Q_3 \big\rangle \end{aligned} \tag{31}$$

where

$$Q_l = \left(\alpha_{lab}u^b + \beta_{la}\right)\partial_{u^a}, \quad \alpha_{lab}, \beta_{la} - \text{const},\ l = \overline{1,3}, \tag{32}$$

and operators Q_l should satisfy the following conditions

$$[Q_1, Q_2] = -Q_1, \qquad [Q_1, Q_3] = 0, \qquad [Q_2, Q_3] = 2Q_3. \tag{33}$$

Table 1
Non-equivalent representations of the extended Galilean algebra without mass operator.

	Q_1	Q_2
1.	∂_{u^1}	$-u^1\partial_{u^1} + ku^2\partial_{u^2}$
2.	∂_{u^1}	$-I + u^2\partial_{u^1}$
3.	∂_{u^1}	$-u^1\partial_{u^1} + \partial_{u^2}$
4.	$\partial_{u^1} + u^1\partial_{u^2}$	$-I - u^2\partial_{u^2}$
5.	$u^1\partial_{u^2}$	$kI + u^1\partial_{u^1}$
6.	$u^1\partial_{u^2}$	$u^1\partial_{u^1} + \partial_{u^2}$

Table 2
Non-equivalent representations of the generalized Galilean algebra without mass operator.

	Q_1	Q_2	Q_3
1.	∂_{u^1}	$-u^1\partial_{u^1} + ku^2\partial_{u^2}$	0
2.	∂_{u^1}	$-u^1\partial_{u^1} + u^2\partial_{u^2}$	$u^2\partial_{u^1}$
3.	∂_{u^1}	$-I + u^2\partial_{u^1}$	0
4.	∂_{u^1}	$-u^1\partial_{u^1} + \partial_{u^2}$	0
5.	$\partial_{u^1} + \tau u^1\partial_{u^2}$	$-I - u^2\partial_{u^2}$	$p\partial_{u^2}$
6.	$u^1\partial_{u^2}$	$-I - u^2\partial_{u^2}$	∂_{u^2}
7.	$u^1\partial_{u^2}$	$kI + u^1\partial_{u^1}$	0
8.	$u^1\partial_{u^2}$	$u^1\partial_{u^1} + \partial_{u^2}$	0

4.2. Classification of the representations of the Galilean algebra $AG(1,1)$ and its extensions

In [18] the representations of the Galilean algebra without mass operator are classified, and it is found that there are 5 different representations of this algebra, which are non-equivalent with respect to transformations (24). In fact, as it can be seen from formula (29), these are unequal representations of the operator Q_1, which we give below, taking into account also transformations (23):

$$Q_1 = \partial_{u^1} + \tau u^1\partial_{u^2}, \qquad Q_1 = \partial_{u^1} + u^2\partial_{u^2}, \qquad Q_1 = \tau I + u^1\partial_{u^2},$$
$$Q_1 = I + ku^2\partial_{u^2}, \qquad Q_1 = kI + J \tag{34}$$

where $\tau \in \{0,1\}$, $k \in \mathbb{R}$, $I = u^1\partial_{u^1} + u^2\partial_{u^2}$, $J = u^1\partial_{u^2} - u^2\partial_{u^1}$.

Thus, the only possible realization of the algebra $AG(1,1)$ for system (1) is algebra (29), where operator Q_1 has the form of one of those five cases presented in formulas (34).

In paper [48] representations of the extended and generalized Galilean algebras without mass operator are classified. In fact, these are non-equivalent sets of operators Q_1, Q_2 for the extended Galilean algebra and operators Q_1, Q_2, Q_3 for the generalized Galilean algebra. In [48] these operators are given to within transformations (24). Since the system (1), besides transformations (24), allows also equivalence transformations (23), so we specify the form of operators Q_l. The results of this specification are represented in Tables 1–2.

Therefore, the only possible realization of the algebra $AG_1(1,1)$ for system (1) is algebra (30), where operators Q_1, Q_2 have one of the six forms shown in Table 1.

The only possible realization of the algebra $AG_2(1,1)$ for system (1) is algebra (31), where operators Q_1, Q_2, Q_3 have one of the eight forms shown in Table 2.

In Table 2 $(\tau, p) \in \{(0,1),(1,k)\}$.

4.3. Representations of the Galilean algebra $AG^M(1,1)$ and its extensions

We call the Galilean algebra with mass operator as one of the realizations of a four-dimensional linear algebra of differential operators of the first order, for which commutation relations (9) and (12) are fulfilled.

Table 3
Non-equivalent representations of the Galilean algebra with mass operator.

	Q_0	Q_1
1.	∂_{u^1}	$\tau_1\partial_{u^2} + \tau_2 u^2\partial_{u^1}$
2.	$r_1\partial_{u^1} + r_2 u^2\partial_{u^2}$	$l_1\partial_{u^1} + l_2 u^2\partial_{u^2}$
3.	$\tau_1\partial_{u^1} + u^1\partial_{u^2}$	$\tau_2\partial_{u^2}$
4.	$s_1 u^1\partial_{u^1} + s_2 u^2\partial_{u^2}$	$p_1 u^1\partial_{u^1} + p_2 u^2\partial_{u^2}$
5.	$r_1 I + r_2 u^2\partial_{u^1}$	$l_1 I + l_2 u^2\partial_{u^1}$
6.	$k_1 I + k_2 J$	$p_1 I + p_2 J$

Using commutation relations (9) and (12) and taking into account results of Theorem 1 we get that algebra $AG^M(1,1)$ has the realization

$$AG^M(1,1) = \langle \partial_0,\ \partial_1,\ G = x_0\partial_1 + x_1 Q_0 + Q_1,\ Q_0\rangle \tag{35}$$

where operators Q_0, Q_1 are set by formulas (32) and satisfy condition $[Q_0, Q_1] = 0$.

As we stated above, in [42] non-equivalent realizations of algebras of the 1–4 orders are specified to within all possible local transformations. Since system (1) does not allow all possible equivalence transformation, but only transformation in the form (23) and (24), the class of operators Q_0, Q_1 for it is much wider. Therefore, among the algebras given in [42], algebra (35) is present, but only at $Q_0 = \partial_{u^1}$, $Q_1 = \partial_{u^2}$.

Thus, the only possible realization of the algebra $AG^M(1,1)$ for system (1) is algebra (35). Similarly, using commutation relations (9)–(11) and (12)–(14), we conclude that the basic generators of the extended and generalized Galilean algebras with mass operator, for which system (1) can be invariant, are

$$AG_1^M(1,1) = \langle \partial_0,\ \partial_1,\ G = x_0\partial_1 + x_1 Q_0 + Q_1,\ Q_0,\ D = 2x_0\partial_0 + x_1\partial_1 + Q_2\rangle, \tag{36}$$

$$AG_2^M(1,1) = \Big\langle \partial_0,\ \partial_1,\ G = x_0\partial_1 + x_1 Q_0 + Q_1,\ Q_0,\ D = 2x_0\partial_0 + x_1\partial_1 + Q_2,$$
$$\Pi = x_0^2\partial_0 + x_0 x_1\partial_1 + x_0 Q_2 + x_1 Q_1 + \frac{x_1^2}{2} Q_0 + Q_3\Big\rangle \tag{37}$$

where $Q_\nu = (\alpha_{\nu ab}u^b + \beta_{\nu a})\partial_{u^a}$, $\alpha_{\nu ab}$, $\beta_{\nu a}$ are constants, $\nu = \overline{0,3}$, and operators Q_ν must satisfy conditions (33) and the following commutation relations: $[Q_0, Q_l] = 0$.

4.4. Classification of the representations of the Galilean algebra $AG^M(1,1)$ and its extensions

The classification of the representations of the Galilean algebra with mass operator, and the extended and generalized Galilean algebra with mass operator is made in paper [48], but this classification, as we have noted above, was made only to within equivalence transformations (24). Therefore, we specify representations of operators Q_ν by using transformations (23). Furthermore, considering the fact that Q_0 is a part of $AG^M(1,1)$, $AG_1^M(1,1)$ and $AG_2^M(1,1)$, we used it to reduce the representations of operators Q_l, taking a linear combination of operator Q_0 with appropriate operators of these algebras. The results of the reduction are given in Tables 3, 4 and 5. Thus, non-equivalent sets of operators Q_0, Q_1, which are possible to within equivalence transformations (23), (24), are represented in Table 3.

In Table 3 $(r_1, r_2) \in \{(0,1), (1,0), (1,1)\}$, and

at $(r_1, r_2) = (0,1)$, $(l_1, l_2) = (1,0)$,
at $(r_1, r_2) = (1,0)$, $(l_1, l_2) = (0,1)$,
at $(r_1, r_2) = (1,1)$, $(l_1, l_2) \in \{(0,0), (0,1)\}$;

$(s_1, s_2) \in \{(1,s), (0,1)\}$, and

Table 4
Non-equivalent representations of the extended Galilean algebra with mass operator.

	Q_0	Q_1	Q_2
1.	∂_{u^1}	0	$\tau_1\partial_{u^2} + \tau_2 u^2\partial_{u^1}$
2.	∂_{u^1}	$\tau u^2\partial_{u^1}$	$qu^2\partial_{u^2} + k_3\partial_{u^1}$
3.	∂_{u^1}	∂_{u^2}	$k_3\partial_{u^1} - u^2\partial_{u^2}$
4.	$\tau\partial_{u^1} + u^2\partial_{u^2}$	0	$p\partial_{u^1} + k_3 u^2\partial_{u^2}$
5.	$u^2\partial_{u^2}$	∂_{u^1}	$k_3 u^2\partial_{u^2} - u^1\partial_{u^1}$
6.	$\tau\partial_{u^1} + u^1\partial_{u^2}$	0	$p\partial_{u^2} + k_3(\tau\partial_{u^1} + u^1\partial_{u^2})$
7.	$u^1\partial_{u^2}$	∂_{u^2}	$-I$
8.	$s_1u^1\partial_{u^1} + s_2u^2\partial_{u^2}$	0	$m_1u^1\partial_{u^1} + m_2u^2\partial_{u^2}$
9.	$\tau I + pu^2\partial_{u^1}$	0	$n_1I + n_2u^2\partial_{u^1}$
10.	I	$u^2\partial_{u^1}$	$u^2\partial_{u^2} + k_3I$
11.	$\tau I + pJ$	0	$m_1I + m_2J$

at $(s_1, s_2) = (1, s)$, $(p_1, p_2) \in \{(0, 0), (0, 1)\}$,
at $(s_1, s_2) = (0, 1)$, $(p_1, p_2) \in \{(0, 0), (1, 0)\}$;

$(k_1, k_2) \in \{(1, k), (0, 1)\}$, and

at $(k_1, k_2) = (1, k)$, $(p_1, p_2) \in \{(0, 0), (0, 1)\}$,
at $(k_1, k_2) = (0, 1)$, $(p_1, p_2) \in \{(0, 0), (1, 0)\}$,

$k, k_3, s \in \mathbb{R}$, and $s \neq 0$, $k_2^2 + p_2^2 \neq 0$;

Thus, to within equivalence transformations (23) and (24) the only possible realization of the Galilean algebra with mass operator for system (1) is algebra (35), where operators Q_0, Q_1 have one of the six forms shown in Table 3.

Similarly, to within equivalence transformations (23) and (24), for system (1) the only possible realization of the extended Galilean algebra with mass operator is algebra (36), where operators Q_0, Q_1, Q_2 have one of the forms given in Table 4.

In Table 4 $(\tau, q) \in \{(0, s), (1, 1)\}$, $(\tau, p) \in \{(1, k), (0, 1)\}$, and

if $(\tau, p) = (1, k)$, then $(n_1, n_2) = (0, 1)$, $(m_1, m_2) = \{(0, m)\}$,
if $(\tau, p) = (0, 1)$, then $(n_1, n_2) = (m, 0)$, $(m_1, m_2) = \{(m, 0)\}$;

$(s_1, s_2) \in \{(1, s), (0, 1)\}$, and

at $(s_1, s_2) = (1, s)$, $(m_1, m_2) = \{(0, m)\}$,
at $(s_1, s_2) = (0, 1)$, $(m_1, m_2) = \{(m, 0)\}$;

$m \in \mathbb{R}$.

All the possible sets of operators Q_0, Q_1, Q_2, Q_3 for algebra (37), which are non-equivalent under equivalence transformations (23), (24), are represented in Table 5.

Thus, to within equivalence transformations (23) and (24) the only possible realization of the generalized Galilean algebra with mass operator for system (1) is algebra (37), where operators Q_0, Q_1, Q_2, Q_3 have one of the forms shown in Table 5.

5. Invariance of system (1) under the Galilean algebras

5.1. Invariance of system (1) under the Galilean algebra without mass operator

We study with what nonlinearities F, G and H, system (1) is invariant under the Galilean algebra $AG(1, 1)$.

Table 5
Non-equivalent representations of the generalized Galilean algebra with mass operator.

	Q_0	Q_1	Q_2	Q_3
1.	∂_{u^1}	0	$\tau_1\partial_{u^2}+\tau_2 u^2\partial_{u^1}$	0
2.	∂_{u^1}	0	$k_3\partial_{u^1}-2u^2\partial_{u^2}$	$\tau\partial_{u^2}$
3.	∂_{u^1}	0	$k_3\partial_{u^1}+2u^2\partial_{u^2}$	$\tau u^2\partial_{u^1}$
4.	∂_{u^1}	$\tau u^2\partial_{u^1}$	$k_3\partial_{u^1}+u^2\partial_{u^2}$	0
5.	∂_{u^1}	$\tau\partial_{u^2}$	$k_3\partial_{u^1}-u^2\partial_{u^2}$	0
6.	$\tau\partial_{u^1}+u^2\partial_{u^2}$	0	$k_3u^2\partial_{u^2}+p\partial_{u^1}$	0
7.	$u^2\partial_{u^2}$	0	$k_3u^2\partial_{u^2}-2u^1\partial_{u^1}$	$\tau\partial_{u^1}$
8.	$\tau\partial_{u^1}+u^1\partial_{u^2}$	0	$k_3(\tau\partial_{u^1}+u^1\partial_{u^2})+p\partial_{u^2}$	0
9.	$u^1\partial_{u^2}$	0	$k_3u^1\partial_{u^2}-2I$	$\tau\partial_{u^2}$
10.	$u^1\partial_{u^2}$	∂_{u^2}	$-I$	0
11.	$s_1u^1\partial_{u^1}+s_2u^2\partial_{u^2}$	0	$m_1u^1\partial_{u^1}+m_2u^2\partial_{u^2}$	0
12.	$u^2\partial_{u^2}$	$\tau\partial_{u^1}$	$k_3u^2\partial_{u^2}-u^1\partial_{u^1}$	0
13.	I	$u^2\partial_{u^1}$	$k_3I+u^2\partial_{u^2}$	0
14.	I	0	$k_3I-2u^2\partial_{u^2}$	$\tau u^1\partial_{u^2}$
15.	I	0	$k_3I+2u^2\partial_{u^2}$	$\tau u^2\partial_{u^1}$
16.	$\tau I+pu^2\partial_{u^1}$	0	$n_1I+n_2u^2\partial_{u^1}$	0
17.	$\tau I+pJ$	0	m_1I+m_2J	0

The following statement is true.

Theorem 2. *System (1) is invariant under Galilean algebra (29) if and only if nonlinearities $F(U)$, $G(U)$ and $H(U)$, to within equivalence transformations (23) and (24), are*

1. $$F(U)=D(u^1,\varphi)=\begin{pmatrix}\varphi^{11}-\varphi^{12}\tau u^1 & \varphi^{12}\\ \varphi^{21}+(\varphi^{11}-\varphi^{22})\tau u^1-\varphi^{12}(\tau u^1)^2 & \varphi^{22}+\varphi^{12}\tau u^1\end{pmatrix},$$
$$G(U)=D(u^1,\psi)-u^1E,\qquad H(U)=\begin{pmatrix}\chi^1\\ \chi^2+\chi^1\tau u^1\end{pmatrix},\qquad \omega=u^2-\tau\frac{(u^1)^2}{2},$$
if $Q_1=\partial_{u^1}+\tau u^1\partial_{u^2}$;

2. $$F(U)=D(u^2,\varphi)=\begin{pmatrix}\varphi^{11} & \frac{\varphi^{12}}{u^2}\\ \varphi^{21}u^2 & \varphi^{22}\end{pmatrix},$$
$$G(U)=D(u^2,\psi)-u^1E,\qquad H(U)=\begin{pmatrix}\chi^1\\ \chi^2u^2\end{pmatrix},\qquad \omega=u^2e^{-u^1},$$
if $Q_1=\partial_{u^1}+u^2\partial_{u^2}$;

3. $$F(U)=D\left(\frac{u^2}{u^1},\varphi\right)=\begin{pmatrix}\varphi^{11}-\varphi^{12}\frac{u^2}{u^1} & \varphi^{12}\\ \varphi^{21}+(\varphi^{11}-\varphi^{22})\frac{u^2}{u^1}-\varphi^{12}(\frac{u^2}{u^1})^2 & \varphi^{22}+\varphi^{12}\frac{u^2}{u^1}\end{pmatrix},$$
$$G(U)=D\left(\frac{u^2}{u^1},\psi\right)-\frac{u^2}{u^1}E,\qquad H(U)=\begin{pmatrix}\chi^1u^1\\ \chi^1u^2+\chi^2u^1\end{pmatrix},\qquad \omega=u^1e^{-\tau\frac{u^2}{u^1}},$$
if $Q_1=\tau I+u^1\partial_{u^2}$;

4. $$F(U)=D\left(\frac{u^2}{u^1},\varphi\right)=\begin{pmatrix}\varphi^{11} & \varphi^{12}\frac{u^1}{u^2}\\ \varphi^{21}\frac{u^2}{u^1} & \varphi^{22}\end{pmatrix},$$
$$G(U)=D\left(\frac{u^2}{u^1},\psi\right)-\ln u^1E,\qquad H(U)=\begin{pmatrix}\chi^1u^1\\ \chi^2u^2\end{pmatrix},\qquad \omega=\frac{(u^1)^{k+1}}{u^2},$$
if $Q_1=I+ku^2\partial_{u^2}$;

5. $$F(U) = D\big(u^1, u^2, \varphi\big) = \begin{pmatrix} \varphi^3 - \frac{2u^1}{\vec{u}^2}\varepsilon_{ab}\varphi^a u^b & -\varphi^4 - \frac{2u^2}{\vec{u}^2}\varepsilon_{ab}\varphi^a u^b \\ \varphi^4 - \frac{2u^1}{\vec{u}^2}\delta_{ab}\varphi^a u^b & \varphi^3 - \frac{2u^2}{\vec{u}^2}\delta_{ab}\varphi^a u^b \end{pmatrix},$$

$$G(U) = D\big(u^1, u^2, \psi\big) - \arctan\frac{u^1}{u^2}E, \qquad H(U) = \begin{pmatrix} \delta_{ab}\chi^a u^b \\ -\varepsilon_{ab}\chi^a u^b \end{pmatrix}, \qquad \omega = \vec{u}^2 e^{2k\arctan\frac{u^1}{u^2}},$$

if $Q_1 = kI + J$,

where $\vec{u} = (u^1, u^2)$, $\varphi^{ab} = \varphi^{ab}(\omega)$, $\psi^{ab} = \psi^{ab}(\omega)$, $\chi^a = \chi^a(\omega)$, $\varphi^i = \varphi^i(\omega)$, $\psi^i = \psi^i(\omega)$ *are arbitrary smooth functions of argument* ω, *view of which is given in every item,* E *is a* 2×2 *unit matrix,* $\varepsilon = (\varepsilon_{ab}) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $i = \overline{1,4}$.

Proof. We find functional matrices $F(U)$, $G(U)$ and $H(U)$, at which system (1) is invariant under Galilean algebra (29). For this we have to substitute corresponding functions ξ^μ and η^a, which were obtained from the representations of the operators of algebra (29), into system of determining equations (16)–(20) from Lemma 1 and solve it for each realization in (34). Since formula (27) specifies the form of infinitesimal operator (15), satisfying Eqs. (16) and (17) of the determining system, we substitute corresponding functions ξ^μ and η^a already only into Eqs. (18)–(20) of the determining system.

We illustrate the proof of this theorem by the example of the first realization in (34). It determines the basic operators of the Galilean algebra

$$\partial_0, \quad \partial_1, \quad G = x_0\partial_1 + \partial_{u^1} + \tau u^1\partial_{u^2} \tag{38}$$

and the coordinates of operator X:

$$\xi^0 = d_0, \qquad \xi^1 = gx_0 + d_1, \qquad \eta^1 = g, \qquad \eta^2 = g\tau u^1 \tag{39}$$

where g, d_μ are arbitrary constants. Substituting (39) into system (18)–(20) we see that it is reduced to the following system for finding functions f^{ab}, g^{ab} and h^a and for specifying the coordinates of the infinitesimal operator:

$$\big(\delta_{c1} + \delta_{c2}\tau u^1\big)f^{ab}_{u^c} + \tau\big(\delta_{b1}f^{a2} - \delta_{a2}f^{1b}\big) = 0, \tag{40}$$

$$\big(\delta_{c1} + \delta_{c2}\tau u^1\big)g^{ab}_{u^c} + \tau\big(\delta_{b1}g^{a2} - \delta_{a2}g^{1b}\big) + \delta_{ab} = 0, \tag{41}$$

$$\big(\delta_{c1} + \delta_{c2}\tau u^1\big)h^a_{u^c} - \delta_{a2}\tau h^1 = 0. \tag{42}$$

The general solution of system (40)–(42) is:

$$\begin{aligned}
f^{11} &= \varphi^{11} - \varphi^{12}\tau u^1, & f^{12} &= \varphi^{12}, \\
f^{21} &= \varphi^{21} + \big(\varphi^{11} - \varphi^{22}\big)\tau u^1 - \varphi^{12}\big(\tau u^1\big)^2, & f^{22} &= \varphi^{22} + \varphi^{12}\tau u^1; \\
g^{11} &= \psi^{11} - \big(1 + \tau\psi^{12}\big)u^1, & g^{12} &= \psi^{12}, \\
g^{21} &= \psi^{21} + \big(\psi^{11} - \psi^{22}\big)\tau u^1 - \psi^{12}\big(\tau u^1\big)^2, & g^{22} &= \psi^{22} - \big(1 - \tau\psi^{12}\big)u^1; \\
h^1 &= \chi^1, & h^2 &= \chi^2 + \chi^1\tau u^1,
\end{aligned}$$

under which system (1) is invariant under the Galilean algebra without mass operator. Here, $\varphi^{ab} = \varphi^{ab}(\omega)$, $\psi^{ab} = \psi^{ab}(\omega)$ and $\chi^a = \chi^a(\omega)$ are arbitrary smooth functions and $\omega = u^2 - \tau\frac{(u^1)^2}{2}$, that coincides with the first item of Theorem 2.

Solving in the similar way system of the determining equations for the other representations of the Galilean algebra without mass operator (34), we obtain the rest of the items of Theorem 2.

The theorem is proved. □

5.2. Invariance of system (1) under the extended Galilean algebra without mass operator

We study at what nonlinearities F, G and H system (1) is invariant under the algebra $AG_1(1,1)$. The following statement is true.

Theorem 3. *System (1) is invariant under extended Galilean algebra (30) if and only if nonlinearities F, G and H, to within equivalence transformations (23) and (24), have the form*

1. $$F(U) = D(u^2,\lambda) = \begin{pmatrix} \lambda_{11} & \lambda_{12}(u^2)^{m-1} \\ \lambda_{21}(u^2)^{1-m} & \lambda_{22} \end{pmatrix},$$
$$G(U) = D(u^2,\mu)(u^2)^m - u^1 E, \qquad H(U) = \begin{pmatrix} \nu_1 (u^2)^{3m} \\ \nu_2 (u^2)^{2m+1} \end{pmatrix}$$
when $Q_1 = \partial_{u^1}$, $Q_2 = -u^1\partial_{u^1} - \frac{1}{m}u^2\partial_{u^2}$;

2. $$F(U) = \begin{pmatrix} \varphi^1(u^2) & 0 \\ 0 & \varphi^2(u^2) \end{pmatrix}, \qquad G(U) = \begin{pmatrix} 0 & 0 \\ \psi(u^2) & 0 \end{pmatrix} - u^1 E, \qquad H(U) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
when $Q_1 = \partial_{u^1}$, $Q_2 = -u^1\partial_{u^1}$;

3. $$F(U) = D(u^2,\lambda) = \begin{pmatrix} \lambda_{11} - \lambda_{21}\ln u^2 & \lambda_{12} + (\lambda_{11} - \lambda_{22})\ln u^2 - \lambda_{21}\ln^2 u^2 \\ \lambda_{21} & \lambda_{22} + \lambda_{21}\ln u^2 \end{pmatrix},$$
$$G(U) = D(u^2,\mu)u^2 - (u^1 + u^2\ln u^2)E, \qquad H(U) = \begin{pmatrix} \nu_1 - \nu_2\ln u^2 \\ \nu_2 \end{pmatrix}(u^2)^3$$
when $Q_1 = \partial_{u^1}$, $Q_2 = -I + u^2\partial_{u^1}$;

4. $$F(U) = D(u^2,\lambda) = \begin{pmatrix} \lambda_{11} & \lambda_{12}e^{-u^2} \\ \lambda_{21}e^{u^2} & \lambda_{22} \end{pmatrix},$$
$$G(U) = D(u^2,\mu)e^{-u^2} - u^1 E, \qquad H(U) = \begin{pmatrix} \nu_1 e^{-3u^2} \\ \nu_2 e^{-2u^2} \end{pmatrix}$$
when $Q_1 = \partial_{u^1}$, $Q_2 = -u^1\partial_{u^1} + \partial_{u^2}$;

5. $$F(U) = D(u^1,\omega,\lambda) = \begin{pmatrix} \lambda_{11} - \lambda_{12}\frac{u^1}{\sqrt{\omega}} & \frac{\lambda_{12}}{\sqrt{\omega}} \\ \lambda_{21}\sqrt{\omega} + (\lambda_{11} - \lambda_{22})u^1 - \lambda_{12}\frac{(u^1)^2}{\sqrt{\omega}} & \lambda_{22} + \lambda_{12}\frac{u^1}{\sqrt{\omega}} \end{pmatrix},$$
$$G(U) = D(u^1,\omega,\mu)\sqrt{\omega} - u^1 E, \qquad H(U) = \begin{pmatrix} \nu_1 \\ \nu_1 u^1 + \nu_2\sqrt{\omega} \end{pmatrix}\omega^{\frac{3}{2}},$$
$\omega = u^2 - \frac{(u^1)^2}{2}$, *when* $Q_1 = \partial_{u^1} + u^1\partial_{u^2}$, $Q_2 = -I - u^2\partial_{u^2}$;

6. $$F(U) = D(u^1,u^2,\lambda) = \begin{pmatrix} \lambda_{11} - \lambda_{12}\frac{u^2}{(u^1)^{m+1}} & \lambda_{12}(u^1)^{-m} \\ \lambda_{21}(u^1)^m + (\lambda_{11} - \lambda_{22})\frac{u^2}{u^1} - \lambda_{12}\frac{(u^2)^2}{(u^1)^{m+2}} & \lambda_{22} + \lambda_{12}\frac{u^2}{(u^1)^{m+1}} \end{pmatrix},$$
$$G(U) = D(u^1,u^2,\mu)(u^1)^m - \frac{u^2}{u^1}E, \qquad H(U) = \begin{pmatrix} \nu_1 u^1 \\ \nu_2 (u^1)^{m+1} + \nu_1 u^2 \end{pmatrix}(u^1)^{2m}$$
when $Q_1 = u^1\partial_{u^2}$, $Q_2 = u^1\partial_{u^1} - \frac{m+1}{m}I$;

7. $$F(U) = \begin{pmatrix} \varphi^1(u^1) & 0 \\ (\varphi^1(u^1) - \varphi^2(u^1))\frac{u^2}{u^1} & \varphi^2(u^1) \end{pmatrix},$$

$$G(U)=\begin{pmatrix}-1 & \frac{u^1}{u^2}\\ -\frac{u^2}{u^1} & 1\end{pmatrix}\frac{u^2}{u^1}\psi(u^1)-\frac{u^2}{u^1}E,\qquad H(U)=\begin{pmatrix}0\\0\end{pmatrix}$$

when $Q_1=u^1\partial_{u^2}$, $Q_2=-u^2\partial_{u^2}$;

8. $$F(U)=D(u^1,u^2,\lambda)=\begin{pmatrix}\lambda_{11}-\lambda_{12}(u^2-\ln u^1) & \lambda_{12}u^1\\ \frac{\lambda_{21}+(\lambda_{11}-\lambda_{22})(u^2-\ln u^1)-\lambda_{12}(u^2-\ln u^1)^2}{u^1} & \lambda_{22}+\lambda_{12}(u^2-\ln u^1)\end{pmatrix},$$

$$G(U)=D(u^1,u^2,\mu)\frac{1}{u^1}-\frac{u^2-\ln u^1}{u^1}E,\qquad H(U)=\begin{pmatrix}\nu_1u^1\\ \nu_2+\nu_1(u^2-\ln u^1)\end{pmatrix}(u^1)^{-2}$$

when $Q_1=u^1\partial_{u^2}$, $Q_2=u^1\partial_{u^1}+\partial_{u^2}$,

where φ^a *and* ψ *are arbitrary smooth functions of their arguments and* λ_{ab}, μ_{ab}, ν_a *and* m *are arbitrary constants.*

Proof. We find functional matrices $F(U)$, $G(U)$ and $H(U)$, at which ones system (1) is invariant under extended Galilean algebra (30). To do this, we use the classification of representations of algebra (30) shown in Table 1.

Since Eqs. (16)–(17) of the determining system have been solved (see Theorem 1), so, for finding functions $f^{ab}(u)$, $g^{ab}(u)$ and $h^a(u)$, at which ones the system of equations (1) is invariant under the extended Galilean algebra without mass operator, we substitute corresponding functions ξ^μ and η^a, which were got from the representations of operators of algebra (30) into Eqs. (18)–(20) of the determining system and solve them for each item of Table 1.

We illustrate the proof of this theorem by the example of a realization of the extended Galilean algebra, which was obtained in the first item of Table 1. The realization of operators Q_1 and Q_2, which are written down in that item, defines the basic operators of the extended Galilean algebra

$$\partial_0,\quad \partial_1,\quad G=x_0\partial_1+\partial_{u^1},\qquad D=2x_0\partial_0+x_1\partial_1-u^1\partial_{u^1}+ku^2\partial_{u^2} \tag{43}$$

and the coordinates of operator X:

$$\xi^0=2cx_0+d_0,\qquad \xi^1=cx_1+gx_0+d_1,\qquad \eta^1=-cu^1+g,\qquad \eta^2=cku^2 \tag{44}$$

where c, g and d_μ are arbitrary parameters.

Substituting (44) into determining system (18)–(20) and taking into account formulas (40)–(42) for $\tau=0$, we see that it is reduced to the following system for finding functional matrices $F(U)$, $G(U)$, $H(U)$:

$$-u^1f^{ab}_{u^1}+ku^2f^{ab}_{u^2}+\delta_{a1}f^{1b}-\delta_{b1}f^{a1}+k(\delta_{b2}f^{a2}-\delta_{a2}f^{2b})=0, \tag{45}$$

$$-u^1g^{ab}_{u^1}+ku^2g^{ab}_{u^2}+\delta_{a1}g^{1b}-\delta_{b1}g^{a1}+k(\delta_{b2}g^{a2}-\delta_{a2}g^{2b})+g^{ab}=0, \tag{46}$$

$$-u^1h^a_{u^1}+ku^2h^a_{u^2}+\delta_{a1}h^1-\delta_{a2}kh^2+2h^a=0. \tag{47}$$

It is evident that algebra (43) is obtained by adding a dilatation operator D to algebra (38) at $\tau=0$. Since the nonlinearities, at which system (1) is invariant under the Galilean algebra without mass operator for $Q_1=\partial_{u^1}$, have been found in Theorem 2, we use them to solve system (45)–(47). We substitute the meaning of the nonlinearities from the first item of Theorem 2 at $\tau=0$:

$$F=\begin{pmatrix}\varphi^{11} & \varphi^{12}\\ \varphi^{21} & \varphi^{22}\end{pmatrix},\qquad G=\begin{pmatrix}\psi^{11} & \psi^{12}\\ \psi^{21} & \psi^{22}\end{pmatrix}-u^1E,\qquad H=\begin{pmatrix}\chi^1\\ \chi^2\end{pmatrix},\qquad \omega=u^2$$

into system (45)–(47) and solve it. The solution of the obtained system obviously depends on the meanings of constant k. Thus, at $k \neq 0$ the general solution of system (45)–(47) is

$$
\begin{aligned}
f^{11} &= \lambda_{11}, & f^{12} &= \lambda_{12}(u^2)^{-\frac{1}{k}-1},\\
f^{21} &= \lambda_{21}(u^2)^{\frac{1}{k}+1}, & f^{22} &= \lambda_{22};\\
g^{11} &= \mu_{11}(u^2)^{-\frac{1}{k}} - u^1, & g^{12} &= \mu_{12}(u^2)^{-\frac{2}{k}-1},\\
g^{21} &= \mu_{21}u^2, & g^{22} &= \mu_{22}(u^2)^{-\frac{1}{k}} - u^1;\\
h^1 &= \nu_1(u^2)^{-\frac{3}{k}}, & h^2 &= \nu_2(u^2)^{-\frac{2}{k}+1}
\end{aligned}
\tag{48}
$$

where λ_{ab}, μ_{ab}, ν_a are arbitrary constants.

Thus, with nonlinearities (48) the system (1) is invariant under the extended Galilean algebra. Substituting $-\frac{1}{k} = m$ into the obtained functions, we proceed to the nonlinearities described in the first item of Theorem 3.

At $k = 0$, system (45)–(47) is satisfied for the functions

$$
F(U) = \begin{pmatrix} \varphi^1 & 0 \\ 0 & \varphi^2 \end{pmatrix}, \qquad G(U) = \begin{pmatrix} -u^1 & 0 \\ \psi & -u^1 \end{pmatrix}, \qquad H(U) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\tag{49}
$$

where $\varphi^a = \varphi^a(u^2)$ and $\psi = \psi(u^2)$ are arbitrary functions, which correspond to the second item of Theorem 3.

Similarly, solving the system of the determining equations for all the presentations of algebra (30) from Table 2, we obtain the rest of the items of Theorem 3.

The theorem is proved. □

5.3. Invariance of system (1) under the generalized Galilean algebra without mass operator

We research which meanings of functional matrices F, G and H allow system (1) to be invariant under the algebra $AG_2(1,1)$.

The following statement is true.

Theorem 4. *System (1) is invariant under generalized Galilean algebra (31) if and only if nonlinearities F, G and H, to within equivalence transformations (23) and (24), have the form*

1.
$$
F(U) = \begin{pmatrix} \lambda_{11} & \lambda_{12}(u^2)^{m-1} \\ 0 & \lambda_{22} \end{pmatrix},
$$
$$
G(U) = \begin{pmatrix} 0 & \mu_{12}(u^2)^{2m-1} \\ -\frac{u^2}{m} & \mu_{22}(u^2)^m \end{pmatrix} - u^1E, \qquad H(U) = \begin{pmatrix} \nu_1(u^2)^{3m} \\ \nu_2(u^2)^{2m+1} \end{pmatrix},
$$
if $Q_1 = \partial_{u^1}$, $Q_2 = -u^1\partial_{u^1} - \frac{1}{m}u^2\partial_{u^2}$, $Q_3 = 0$, $m \neq 0, 1$;

2.
$$
F(U) = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} 0 & \mu_{12} \\ -1 & \mu_{22} \end{pmatrix} u^2 - u^1E, \qquad H(U) = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} (u^2)^3,
$$
if $Q_1 = \partial_{u^1}$, $Q_2 = -I$, $Q_3 = 0$;

3.
$$
F(U) = \begin{pmatrix} \lambda & 0 \\ 0 & \varphi \end{pmatrix}, \qquad G(U) = -u^1E, \qquad H(U) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$
if $Q_1 = \partial_{u^1}$, $Q_2 = -u^1\partial_{u^1}$, $Q_3 = 0$;

4. $$F(U) = \begin{pmatrix} \lambda_{11} & \lambda_{12} + (\lambda_{11} - \lambda_{22}) \ln u^2 \\ 0 & \lambda_{22} \end{pmatrix},$$

$$G(U) = \begin{pmatrix} 1 & \mu_{12} + (1 - \mu_{22}) \ln u^2 + \ln^2 u^2 \\ -1 & \mu_{22} - 2 \ln u^2 \end{pmatrix} u^2 - u^1 E, \qquad H(U) = \begin{pmatrix} \nu_1 - \nu_2 \ln u^2 \\ \nu_2 \end{pmatrix} \left(u^2\right)^3,$$

if $Q_1 = \partial_{u^1}$, $Q_2 = -I + u^2 \partial_{u^1}$, $Q_3 = 0$;

5. $$F(U) = \begin{pmatrix} \lambda_{11} & \lambda_{12} e^{-u^2} \\ 0 & \lambda_{22} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} 0 & \mu_{12} e^{-2u^2} \\ 1 & \mu_{22} e^{-u^2} \end{pmatrix} - u_1 E, \qquad H(U) = \begin{pmatrix} \nu_1 e^{-3u^2} \\ \nu_2 e^{-2u^2} \end{pmatrix},$$

if $Q_1 = \partial_{u^1}$, $Q_2 = -u^1 \partial_{u^1} + \partial_{u^2}$, $Q_3 = 0$;

6. $$F(U) = \begin{pmatrix} -\lambda_{12} u^1 & \lambda_{12} \\ -\lambda_{12}(u^1)^2 - 2\lambda_{12}\omega - \lambda_{22}\sqrt{\omega} u^1 & \lambda_{22}\sqrt{\omega} + \lambda_{12} u^1 \end{pmatrix} \frac{1}{\sqrt{\omega}},$$

$$G(U) = \begin{pmatrix} -\mu_{12} u^1 & \mu_{12} \\ -\mu_{12}(u^1)^2 - 2\omega - \mu_{22}\sqrt{\omega} u^1 & \mu_{22}\sqrt{\omega} + \mu_{12} u^1 \end{pmatrix} - u^1 E,$$

$$H(U) = \begin{pmatrix} \nu_1 \\ \nu_2 \sqrt{\omega} + \nu_1 u^1 \end{pmatrix} \sqrt{\omega^3}, \qquad \omega = u^2 - \frac{(u^1)^2}{2},$$

if $Q_1 = \partial_{u^1} + u^1 \partial_{u^2}$, $Q_2 = -I - u^2 \partial_{u^2}$, $Q_3 = 0$;

7. $$F(U) = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} 0 & \mu_{12} \\ 0 & 0 \end{pmatrix} - u^1 E, \qquad H(U) = \begin{pmatrix} 0 \\ -(u^2)^2 \end{pmatrix},$$

if $Q_1 = \partial_{u^1}$, $Q_2 = -I - u^2 \partial_{u^2}$, $Q_3 = \partial_{u^2}$;

8. $$F(U) = \begin{pmatrix} k(k\nu_2 + 1) & 0 \\ (k(k\nu_2 + 1) - \lambda_{22}) u^1 & \lambda_{22} \end{pmatrix},$$

$$G(U) = \begin{pmatrix} -\mu_{12} u^1 & \mu_{12} \\ -2(k\nu_2 + 1)\omega - \mu_{12}(u^1)^2 & \mu_{12} u^1 \end{pmatrix} - u^1 E, \qquad H(U) = \begin{pmatrix} 0 \\ \nu_2 \omega^2 \end{pmatrix}, \qquad \omega = u^2 - \frac{(u^1)^2}{2},$$

if $Q_1 = \partial_{u^1} + u^1 \partial_{u^2}$, $Q_2 = -I - u^2 \partial_{u^2}$, $Q_3 = k\partial_{u^2}$, $k \neq 0$;

9. $$F(U) = \begin{pmatrix} \lambda_{11} & 0 \\ \lambda_{21}(u^1)^m + 2\lambda_{11} \frac{u^2}{u^1} & -\lambda_{11} \end{pmatrix},$$

$$G(U) = \begin{pmatrix} \mu_{11}(u^1)^m + \frac{1}{m} \frac{u^2}{u^1} & -\frac{1}{m} \\ \mu_{21}(u^1)^{2m} + \mu_{11}(u^1)^m \frac{u^2}{u^1} + \frac{1}{m} (\frac{u^2}{u^1})^2 & -\frac{1}{m} \frac{u^2}{u^1} \end{pmatrix} - \frac{u^2}{u^1} E,$$

$$H(U) = \begin{pmatrix} \nu_1 \\ \nu_2 (u^1)^m + \nu_1 \frac{u^2}{u^1} \end{pmatrix} \left(u^1\right)^{2m+1},$$

if $Q_1 = u^1 \partial_{u^2}$, $Q_2 = -\frac{m+1}{m} I + u^1 \partial_{u^1}$, $Q_3 = 0$, $m \neq 0$;

10. $$F(U) = \begin{pmatrix} \dot{\theta} & 0 \\ \frac{u^2}{u^1} (\frac{\theta}{u^1} + \dot{\theta}) & -\frac{\theta}{u^1} \end{pmatrix}, \qquad G(U) = -\frac{u^2}{u^1} E, \qquad H(U) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

if $Q_1 = u^1 \partial_{u^2}$, $Q_2 = -u^2 \partial_{u^2}$, $Q_3 = 0$;

11. $$F(U) = \begin{pmatrix} \lambda_{11} & 0 \\ \frac{\lambda_{21} + 2\lambda_{11}(u^2 - \ln u^1)}{u^1} & -\lambda_{11} \end{pmatrix},$$

$$G(U) = \begin{pmatrix} \mu_{11} - 2(u^2 - \ln u^1) & u^1 \\ \frac{\mu_{21}+(\mu_{11}-1)(u^2-\ln u^1)-(u^2-\ln u^1)^2}{u^1} & 1 \end{pmatrix} \frac{1}{u^1}, \qquad H(U) = \begin{pmatrix} \nu_1 u^1 \\ \nu_2 + \nu_1(u^2 - \ln u^1) \end{pmatrix} \frac{1}{(u^1)^2},$$

if $Q_1 = u^1\partial_{u^2}$, $Q_2 = u^1\partial_{u^1} + \partial_{u^2}$, $Q_3 = 0$,

where $\theta = \theta(u^1)$, $\varphi = \varphi(u^2)$ *are arbitrary smooth functions of their arguments.*

Proof. We find functional matrices $F(U)$, $G(U)$ and $H(U)$, under which system (1) is invariant under the generalized Galilean algebra without mass operator (31), which is determined by operators Q_1, Q_2 and Q_3. Non-equivalent representations of these operators (i.e. actually non-equivalent representations of the generalized Galilean algebra) are given in Table 2.

As in the previous theorem, for finding functions $f^{ab}(u)$, $g^{ab}(u)$ and $h^a(u)$, which make the system of equations (1) invariant under the generalized Galilean algebra, we substitute the corresponding functions ξ^μ and η^a, derived from the representations of the operators of algebra (31) into Eqs. (18)–(20) of the determining system and solve it for each item in Table 2.

We illustrate the proof of the theorem by the example of the generalized Galilean algebra, which was obtained from the first item of Table 2. The realization of operators Q_1, Q_2 and Q_3, which is written in that item, determines the basic operators of the generalized Galilean algebra:

$$\partial_0, \quad \partial_1, \quad G = x_0\partial_1 + \partial_{u^1}, \qquad D = 2x_0\partial_0 + x_1\partial_1 - u^1\partial_{u^1} + ku^2\partial_{u^2},$$
$$\varPi = x_0^2\partial_0 + x_0x_1\partial_1 + x_1\partial_{u^1} + x_0\big(-u^1\partial_{u^1} + ku^2\partial_{u^2}\big), \quad k \neq -2, \tag{50}$$

and the coordinates of operator X:

$$\xi^0 = ax_0^2 + 2cx_0 + d_0, \qquad \xi^1 = ax_0x_1 + cx_1 + gx_0 + d_1,$$
$$\eta^1 = a\big(x_1 - x_0u^1\big) - cu^1 + g, \qquad \eta^2 = (ax_0 + c)ku^2, \quad k \neq -2 \tag{51}$$

where a, c, g and d_μ are arbitrary parameters.

Substituting (51) into determining system (18)–(20) and taking into account formulas (40)–(42) (at $\tau = 0$) and (45)–(47), we see that it can be reduced to the following system of equations:

$$f^{ab}_{u^1} + f^{a1}_{u^b} = 0, \qquad g^{a1} + \delta_{a1}u^1 - \delta_{a2}ku^2 = 0. \tag{52}$$

It is evident that algebra (50) has been obtained by adding operator of projective transformations $\varPi$ to algebra (43). Since nonlinearities, at which system (1) is invariant under the extended Galilean algebra for $Q_1 = \partial_{u^1}$, $Q_2 = -u^1\partial_{u^1} + ku^2\partial_{u^2}$, have been found in Theorem 3 (namely, functions (48) at $k \neq 0$ and (49) at $k = 0$), we use them to solve system (52).

We start from the case at $k \neq 0$. Substituting functions (48), which were found in Theorem 3, into system (52), we see that its solution depends on the meaning of the constant k as follows:

(a) at $k \neq -1$ the general solution of the system is

$$\begin{aligned} f^{11} &= \lambda_{11}, & f^{12} &= \lambda_{12}\big(u^2\big)^{-\frac{1}{k}-1}, \\ f^{21} &= 0, & f^{22} &= \lambda_{22}; \\ g^{11} &= -u^1, & g^{12} &= \mu_{12}\big(u^2\big)^{-\frac{2}{k}-1}, \\ g^{21} &= ku^2, & g^{22} &= \mu_{22}\big(u^2\big)^{-\frac{1}{k}} - u^1; \\ h^1 &= \nu_1\big(u^2\big)^{-\frac{3}{k}}, & h^2 &= \nu_2\big(u^2\big)^{-\frac{2}{k}+1} \end{aligned} \tag{53}$$

where λ_{ab}, μ_{ab} and ν_a are arbitrary constants.

Substituting $-\frac{1}{k} = m$ into the obtained functions, we proceed to the nonlinearities described in the first item of Theorem 4 at $m \neq 0, 1$.

(b) at $k = -1$ nonlinearities, which satisfy system (52) are

$$F = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & \mu_{12} \\ -1 & \mu_{22} \end{pmatrix} u^2 - u^1 E, \qquad H = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} (u^2)^3$$

where λ_{ab}, μ_{ab} and ν_a are arbitrary constants, which correspond to the second item of Theorem 4.

Now we consider the case at $k = 0$. Substituting functions (49), which were found in Theorem 3 into system (52) and solving it, we can see that at $k = 0$ the system is satisfied with the following values of the nonlinearities:

$$F = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \varphi^{22} \end{pmatrix}, \qquad G = -u^1 E, \qquad H = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \omega = u^2$$

where λ_{11} is an arbitrary constant and φ^{22} is an arbitrary smooth function of argument ω. These nonlinearities coincide with the third item of Theorem 4, where symbols $\lambda_{11} = \lambda$, $\varphi^{22} = \varphi$ are introduced.

Similarly, solving the system of the determining equations for all non-equivalent representations of algebra (31) defined in Table 2, we obtain the rest of the items of Theorem 4.

The theorem is proved. □

5.4. Invariance of system (1) under the Galilean algebra with mass operator

We study with what functional matrices F, G and H system (1) is invariant under the algebra $AG^M(1,1)$. The following statement is true.

Theorem 5. *System of equations (1) is invariant under the Galilean algebra with mass operator (35) if and only if matrices F, G, H, to within equivalence transformations (23) and (24), are as follows:*

$$F(U) = \begin{pmatrix} \lambda_{21}\tau u^2 + \lambda_{11} & -\lambda_{21}(\tau u^2)^2 - (\lambda_{11} - \lambda_{22})\tau u^2 + \lambda_{12} \\ \lambda_{21} & -\lambda_{21}\tau u^2 + \lambda_{22} \end{pmatrix},$$
$$G(U) = \begin{pmatrix} (\tau m_{21} - 1)u^2 + m_{11} & P_2(\tau u^2) \\ m_{21} & -(\tau m_{21} + 1)u^2 + m_{22} \end{pmatrix},$$
$$H(U) = \begin{pmatrix} (-\tau m_{21} + \frac{1}{2})(u^2)^2 + (\tau n_2 - m_{11})u^2 + n_1 \\ -m_{21}u^2 + n_2 \end{pmatrix}, \tag{54}$$

and $Q_0 = \partial_{u^1}$, $Q_1 = \tau u^2 \partial_{u^1} + \partial_{u^2}$, $P_2(\tau u^2) = -m_{21}(\tau u^2)^2 - (m_{11} - m_{22} + \lambda_{21})\tau u^2 + m_{12}$ and λ_{ab}, m_{ab}, n_a are arbitrary constants;

$$F(U) = \begin{pmatrix} \varphi(u^1) & -\frac{u^1}{u^2} \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} \psi^1(u^1) & 0 \\ u^2\psi^2(u^1) & 0 \end{pmatrix}, \qquad H(U) = \begin{pmatrix} \chi^1(u^1) \\ u^2\chi^2(u^1) \end{pmatrix}, \tag{55}$$

and $Q_0 = u^2\partial_{u^2}$, $Q_1 = 0$, $\varphi = \varphi(u^1)$, $\psi^a = \psi^a(u^1)$, $\chi^a = \chi^a(u^1)$ are arbitrary smooth functions;

$$F(U) = \begin{pmatrix} \lambda_{11} & \frac{\lambda_{12}}{u^2} \\ \lambda_{21}u^2 & \lambda_{22} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} -\omega + m_{11} & \frac{m_{12}}{u^2} \\ (-\lambda_{21}\omega + m_{21})u^2 & -(\lambda_{21} + 2\lambda_{22} + 1)\omega + m_{22} \end{pmatrix},$$
$$H(U) = \begin{pmatrix} \frac{\tau}{2}\omega^2 - (\tau m_{11} + m_{12})\omega + n_1 \\ u^2[\frac{1}{2}(2\tau\lambda_{21} + 2\lambda_{22} + 1)\omega^2 - (\tau m_{21} + m_{22})\omega + n_2] \end{pmatrix}, \tag{56}$$

and $Q_0 = \tau\partial_{u^1} + u^2\partial_{u^2}$, $Q_1 = \partial_{u^1}$, $\omega = u^1 - \tau \ln u^2$;

$$F(U) = \begin{pmatrix} \lambda_{11} & \frac{\lambda_{12}}{u^2} \\ \lambda_{21}u^2 & \lambda_{22} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} -\ln u^2 + m_{11} & \frac{m_{12}}{u^2} \\ m_{21}u^2 & -(\lambda_{21}+1)\ln u^2 + m_{22} \end{pmatrix},$$
$$H(U) = \begin{pmatrix} \frac{1}{2}\ln^2 u^2 - m_{11}\ln u^2 + n_1 \\ u^2[-m_{21}\ln u^2 + n_2] \end{pmatrix}, \tag{57}$$

and $Q_0 = \partial_{u^1}$, $Q_1 = u^2\partial_{u^2}$;

$$F(U) = \begin{pmatrix} \lambda_{11} - \lambda_{12}u^1 & \lambda_{12} \\ \lambda_{21} + (\lambda_{11} - \lambda_{22})u^1 - \lambda_{12}(u^1)^2 & \lambda_{22} + \lambda_{12}u^1 \end{pmatrix},$$
$$G(U) = \begin{pmatrix} m_{11} - m_{12}u^1 - \omega & m_{12} \\ P_2(u^1) - (2\lambda_{11} - \lambda_{12}u^1)\omega & m_{22} + m_{12}u^1 - (\lambda_{12}+1)\omega \end{pmatrix},$$
$$H(U) = \begin{pmatrix} \frac{\omega^2}{2} - m_{11}\omega + n_1 \\ (\frac{u^1}{2} + \lambda_{11})\omega^2 - (m_{11}u^1 + m_{21})\omega + n_1u^1 + n_2 \end{pmatrix}, \tag{58}$$

and $Q_0 = \partial_{u^1} + u^1\partial_{u^2}$, $Q_1 = \partial_{u^2}$, $\omega = u^2 - \frac{(u^1)^2}{2}$, $P_2(u^1) = m_{21} + (m_{11} - m_{22})u^1 - m_{12}(u^1)^2$;

$$F(U) = \begin{pmatrix} \lambda_{11} & \lambda_{12}\frac{u^1}{u^2} \\ \lambda_{21}\frac{u^2}{u^1} & \lambda_{22} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} A\ln\omega + m_{11} & (B\ln\omega + m_{12})\frac{u^1}{u^2} \\ (C\ln\omega + m_{21})\frac{u^2}{u^1} & D\ln\omega + m_{22} \end{pmatrix},$$
$$H(U) = \begin{pmatrix} (-\frac{1}{2}(A+sB)\ln^2\omega - (m_{11}+sm_{12})\ln\omega + n_1)u^1 \\ (-\frac{1}{2}(C+sD)\ln^2\omega - (m_{21}+sm_{22})\ln\omega + n_2)u^2 \end{pmatrix}, \tag{59}$$

and $Q_0 = u^1\partial_{u^1} + su^2\partial_{u^2}$, $Q_1 = u^2\partial_{u^2}$, $\omega = \frac{u^2}{(u^1)^s}$, $A = -(s\lambda_{12} + 2\lambda_{11} + 1)$, $B = -\lambda_{12}$, $C = -s\lambda_{21}$, $D = -(\lambda_{21} + 2s\lambda_{22} + 1)$;

$$F(U) = \begin{pmatrix} \lambda_{11} & \lambda_{12}\frac{u^1}{u^2} \\ \lambda_{21}\frac{u^2}{u^1} & \lambda_{22} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} -(\lambda_{12}+1)\ln u^1 + m_{11} & m_{12}\frac{u^1}{u^2} \\ (-\lambda_{21}\ln u^1 + m_{21})\frac{u^2}{u^1} & -(2\lambda_{22}+1)\ln u^1 + m_{22} \end{pmatrix},$$
$$H(U) = \begin{pmatrix} (-m_{12}\ln u^1 + n_1)u^1 \\ (\frac{2\lambda_{22}+1}{2}\ln^2 u^1 - m_{22}\ln u^1 + n_2)u^2 \end{pmatrix}, \tag{60}$$

and $Q_0 = u^2\partial_{u^2}$, $Q_1 = u^1\partial_{u^1}$;

$$F(U) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2s} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} -s\alpha(\omega) & \alpha(\omega)\frac{u^1}{u^2} \\ -s\beta(\omega)\frac{u^2}{u^1} & \beta(\omega) \end{pmatrix}, \qquad H(U) = \begin{pmatrix} \chi^1(\omega)u^1 \\ \chi^2(\omega)u^2 \end{pmatrix}, \tag{61}$$

and $Q_0 = u^1\partial_{u^1} + su^2\partial_{u^2}$, $Q_1 = 0$, $\omega = \frac{u^2}{(u^1)^s}$, α, β, χ^a *are arbitrary smooth functions*, $s \neq 0$;

$$F(U) = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ 0 & \lambda_{11} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} -(2\lambda_{11}+1)\omega + m_{11} & P_1(\omega) + (m_{22} - m_{11})\frac{u^1}{u^2} \\ 0 & -(2\lambda_{11}+1)\omega + m_{22} \end{pmatrix},$$
$$H(U) = \begin{pmatrix} u^2[(\lambda_{12} - \frac{\tau}{2})\omega^2 + \tau m_{11}\omega + n_1] + u^1P_2(\omega) \\ u^2P_2(\omega) \end{pmatrix}, \tag{62}$$

and $Q_0 = I + \tau u^2\partial_{u^1}$, $Q_1 = u^2\partial_{u^1}$, $\omega = \frac{u^1}{u^2} - \tau\ln u^2$, $P_1(\omega) = -2(\tau\lambda_{11} + \lambda_{12})\omega + m_{12}$, $P_2(\omega) = [(\lambda_{11} + \frac{1}{2})\omega^2 - m_{22}\omega + n_2]$;

$$F(U) = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad G(U) = \begin{pmatrix} \psi^{11} + \psi^{21}\frac{u^1}{u^2} & -(\frac{u^1}{u^2}+1)(\psi^{11}+\psi^{21}\frac{u^1}{u^2}) \\ \psi^{21} & -(\frac{u^1}{u^2}+1)\psi^{21} \end{pmatrix},$$
$$H(U) = \begin{pmatrix} u^2\chi^1 + u^1\chi^2 \\ u^2\chi^2 \end{pmatrix}, \tag{63}$$

and $Q_0 = I + u^2\partial_{u^1}$, $Q_1 = 0$, $\omega = \frac{u^1}{u^2} - \ln u^2$, $\psi^{a1} = \psi^{a1}(\omega)$, $\chi^a = \chi^a(\omega)$ *are arbitrary smooth functions;*

$$F(U) = \begin{pmatrix} \lambda_{21}\frac{u^1}{u^2} + \lambda_{11} & -2\lambda(\frac{u^1}{u^2})^2 + (\lambda_{22}-\lambda_{11})\frac{u^1}{u^2} + \lambda_{12} \\ \lambda_{21} & -\lambda_{21}\frac{u^1}{u^2} + \lambda_{22} \end{pmatrix},$$
$$G(U) = \begin{pmatrix} m_{21}\frac{u^1}{u^2} + A & P_2(\frac{u^1}{u^2}) - (\lambda_{11}+\lambda_{22})\ln u^2 \\ m_{21} & -m_{21}\frac{u^1}{u^2} + B \end{pmatrix},$$
$$H(U) = \begin{pmatrix} u^1 P_1(\ln u^2) + u^2[(\lambda_{21}+\frac{1}{2})\ln^2 u^2 - m_{11}\ln u^2 + n_1] \\ u^2 P_1(\ln u^2) \end{pmatrix}, \tag{64}$$

and $Q_0 = u^2\partial_{u^1}$, $Q_1 = I$, $A = -(2\lambda_{21}+1)\ln u^2 + m_{11}$, $B = -(\lambda_{21}+1)\ln u^2 + m_{22}$, $P_2(\frac{u^1}{u^2}) = -m_{21}(\frac{u^1}{u^2})^2 + (B-A)\frac{u^1}{u^2} + m_{12}$, $P_1(\ln u^2) = -m_{21}\ln u^2 + n_2$;

$$F(U) = \begin{pmatrix} c_1 + \lambda_1 - \frac{2u^1}{\vec{u}^2}\vec{\lambda}\vec{u} & c_2 + \lambda_2 - \frac{2u^2}{\vec{u}^2}\vec{\lambda}\vec{u} \\ -c_2 + \lambda_2 - \frac{2u^2}{\vec{u}^2}\vec{\lambda}\vec{u} & c_1 - \lambda_1 + \frac{2u^1}{\vec{u}^2}\vec{\lambda}\vec{u} \end{pmatrix},$$
$$G(U) = \begin{pmatrix} \alpha^1 + \beta^1 - \frac{2u^1}{\vec{u}^2}\vec{\beta}\vec{u} & \alpha^2 + \beta^2 - \frac{2u^2}{\vec{u}^2}\vec{\beta}\vec{u} \\ -\alpha^2 + \beta^2 - \frac{2u^2}{\vec{u}^2}\vec{\beta}\vec{u} & \alpha^1 - \beta^1 + \frac{2u^1}{\vec{u}^2}\vec{\beta}\vec{u} \end{pmatrix}, \qquad H(U) = \begin{pmatrix} u^1\chi^1 + u^2\chi^2 \\ -u^1\chi^2 + u^2\chi^1 \end{pmatrix}, \tag{65}$$

and $Q_0 = k_1 I - k_2 J$, $Q_1 = m_1 I - m_2 J$, $\vec{k}^2 = 1$, $|m_1| + |m_2| \neq 0$, $\omega = k_2 \ln \vec{u}^2 + 2k_1 \operatorname{arctg}\frac{u^2}{u^1}$, $\vec{\alpha} = \vec{p}\omega + \vec{m}$, $\vec{\beta} = \vec{r}\omega + \vec{l}$, $\vec{\chi} = \vec{\gamma}\omega^2 + \vec{\sigma}\omega + \vec{n}$; $\vec{p} = \frac{1}{2\vec{k}\vec{m}^\perp}(2(c_1k_1 - c_2k_2) - \vec{\lambda}\vec{k} + 1, 2(c_1k_2 + c_2k_1) - \vec{\lambda}\vec{k}^\perp)$, $\vec{m} = (m_1, m_2)$, $\vec{r} = \frac{1}{2\vec{k}\vec{m}^\perp}(k_1\lambda_1 - k_2\lambda_2, k_1\lambda_2 + k_2\lambda_1)$, $\vec{n} = (n_1, n_2)$, $\vec{l} = (l_1, l_2)$, $\vec{k}^\perp = (-k^2, k^1)$, $\vec{\gamma} = \frac{k_1\vec{p} - k_1\vec{r} + k_2\vec{p}^\perp + k_2\vec{r}^\perp}{4\vec{k}\vec{m}^\perp}$, $\vec{\sigma} = \frac{k_1\vec{m} - k_1\vec{l} + k_2\vec{m}^\perp + k_2\vec{l}^\perp}{2\vec{k}\vec{m}^\perp}$, $c_a, k_a, \lambda_a, m_a, n_a, l_a \in \mathbb{R}$;

$$F(U) = -\frac{1}{2}\begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix}, \qquad G(U) = \begin{pmatrix} 2k_1\vec{k}\vec{\beta} - \frac{2u^1}{\vec{u}^2}\vec{\beta}\vec{u} & 2k_1\vec{k}^\perp\vec{\beta} - \frac{2u^2}{\vec{u}^2}\vec{\beta}\vec{u} \\ 2k_2\vec{k}\vec{\beta} - \frac{2u^2}{\vec{u}^2}\vec{\beta}\vec{u} & 2k_2\vec{k}^\perp\vec{\beta} + \frac{2u^1}{\vec{u}^2}\vec{\beta}\vec{u} \end{pmatrix},$$
$$H(U) = \begin{pmatrix} u^1\chi^1 + u^2\chi^2 \\ -u^1\chi^2 + u^2\chi^1 \end{pmatrix}, \tag{66}$$

and $Q_0 = k_1 I - k_2 J$, $Q_1 = 0$, $\omega = k_2 \ln \vec{u}^2 + 2k_1 \operatorname{arctg}\frac{u^2}{u^1}$, $\alpha^a = \alpha^a(\omega)$, $\beta^a = \beta^a(\omega)$, $\chi^a = \chi^a(\omega)$ *are arbitrary smooth functions,* k_a *are arbitrary constants and* $\vec{k}^2 = 1$.

Proof. We consider the case when operators Q_0 and Q_1 are $Q_0 = \partial_{u^1}$, $Q_1 = \tau_1\partial_{u^2} + \tau_2 u^2\partial_{u^1}$ (that corresponds to the first item in Table 3). Then the Galilean operator has the following coordinates

$$\xi^0 = 0, \qquad \xi^1 = x_0, \qquad \eta^a = (x_1 + \tau_2 u^2)\delta_{a1} + \tau_1\delta_{a2}. \tag{67}$$

Substituting (67) into (16)–(20) and splitting x_1 into degrees, we obtain

$$f^{ab}_{u^1} = g^{ab}_{u^1} = h^a_{u^1} = 0, \tag{68}$$
$$\tau_1 f^{ab}_{u^2} = \tau_2(f^{2b}\delta_{a1} - f^{a1}\delta_{b2}), \tag{69}$$
$$\tau_1 g^{ab}_{u^2} = \tau_2(g^{2b}\delta_{a1} - g^{a1}\delta_{b2}) - f^{a1}_{u^b} - \delta_{ab}, \tag{70}$$
$$\tau_1 h^a_{u^2} = \tau_2 h^2\delta_{a1} - g^{a1}. \tag{71}$$

Since at $\tau_1 = 0$ system (68)–(71) is inconsistent, we assume that $\tau_1 = 1$.

Thus, without loss of generality, we can assume that

$$G = x_0\partial_1 + x_1\partial_{u^1} + \tau u^2\partial_{u^1} + \partial_{u^2}.$$

The general solution of (68)–(69) is

$$\begin{gathered} f^{11} = \lambda_{21}\tau u^2 + \lambda_{11}, \qquad f^{12} = -\lambda_{21}\left(\tau u^2\right)^2 - (\lambda_{11} - \lambda_{22})\tau u^2 + \lambda_{12}, \\ f^{21} = \lambda_{21}, \qquad f^{22} = -\lambda_{21}\tau u^2 + \lambda_{22}. \end{gathered} \tag{72}$$

Substituting $g^{ab} = \psi^{ab} - u^2\delta_{ab}$ into system (70), we reduce it to (69). Therefore, we can conclude that

$$\begin{gathered} \psi^{11} = m_{21}\tau u^2 + m_{11}, \qquad \psi^{12} = -m_{21}\left(\tau u^2\right)^2 - (m_{11} - m_{22})\tau u^2 + m_{12}, \\ \psi^{21} = m_{21}, \qquad \psi^{22} = -m_{21}\tau u^2 + m_{22}. \end{gathered}$$

After the inverse substitution, we obtain

$$\begin{gathered} g^{11} = (\tau m_{21} - 1)u^2 + m_{11}, \qquad g^{12} = -m_{21}\left(\tau u^2\right)^2 - (m_{11} - m_{22})\tau u^2 + m_{12}, \\ g^{21} = m_{21}, \qquad g^{22} = -(\tau m_{21} + 1)u^2 + m_{22}. \end{gathered} \tag{73}$$

Substituting (73) into the system of equations (71) and solving it, we have

$$h^1 = \left(-\tau m_{21} + \frac{1}{2}\right)\left(u^2\right)^2 + (\tau n_2 - m_{11})u^2 + n_1, \qquad h^2 = -m_{21}u^2 + n_2. \tag{74}$$

Formulas (72), (73) and (74) determine functions (54).

The other items of this theorem are proved in the analogous way.

The theorem is proved. □

5.5. Invariance of system (1) under the extended Galilean algebra with mass operator

We study which functional matrices F, G and H allow system (1) to be invariant under the algebra $AG_1^M(1,1)$.

The following statement is true.

Theorem 6. *System of equations (1) is invariant under extended Galilean algebra with mass operator (36) if and only if matrices F, G and H, to within equivalence transformations (23) and (24), are as follows:*

$$F(U) = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} -u^2 & m_{12} \\ 0 & -u^2 \end{pmatrix}, \qquad H(U) = \frac{1}{2}\left(u^2\right)^2\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{75}$$

and $Q_0 = \partial_{u^1}$, $Q_1 = \partial_{u^2}$, $Q_2 = -u^2\partial_{u^2}$;

$$F(U) = \begin{pmatrix} \lambda_{11} & -\frac{u^1}{u^2} \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad G(U) = \left(u^1\right)^k\begin{pmatrix} m_{11}u^1 & 0 \\ m_{21}u^2 & 0 \end{pmatrix}, \qquad H(U) = \left(u^1\right)^{2(k+1)}\begin{pmatrix} n_1u^1 \\ n_2u^2 \end{pmatrix}, \tag{76}$$

and $Q_0 = u^2\partial_{u^2}$, $Q_1 = 0$, $Q_2 = -\frac{1}{k+1}u^1\partial_{u^1}$, $k \neq -1$;

$$\begin{gathered} F(U) = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix}, \qquad G(U) = \begin{pmatrix} -u^1 & 0 \\ m_{21}u^2 & -(2\lambda_{22} + 1)u^1 \end{pmatrix}, \\ H(U) = \left(u^1\right)^2u^2\begin{pmatrix} 0 \\ \lambda_{22} + \frac{1}{2} \end{pmatrix}, \end{gathered} \tag{77}$$

and $Q_0 = u^2\partial_{u^2}$, $Q_1 = \partial_{u^1}$, $Q_2 = -u^1\partial_{u^1}$;

$$F(U) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2s} \end{pmatrix}, \qquad G(U) = \omega^k \begin{pmatrix} -sm_{11} & m_{11}\frac{u^1}{u^2} \\ -sm_{12}\frac{u^2}{u^1} & m_{12} \end{pmatrix}, \qquad H(U) = \omega^{2k}\begin{pmatrix} n_1u^1 \\ n_2u^2 \end{pmatrix}, \tag{78}$$

and $Q_0 = u^1\partial_{u^1} + su^2\partial_{u^2}$, $Q_1 = 0$, $Q_2 = -\frac{1}{k}u^2\partial_{u^2}$, $\omega = \frac{u^2}{(u^1)^s}$, $s \neq 0$, $k \neq 0$;

$$F(U) = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{11} \end{pmatrix}, \qquad G(U) = -(2\lambda_{11}+1)\frac{u^1}{u^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$H(U) = \left(\lambda_{11} + \frac{1}{2}\right)\left(\frac{u^1}{u^2}\right)^2\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \tag{79}$$

and $Q_0 = I$, $Q_1 = u^2\partial_{u^1}$, $Q_2 = u^2\partial_{u^2}$;

$$F(U) = -\frac{1}{2}\begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix}, \qquad G(U) = e^{m\omega}\begin{pmatrix} m_{11} + m_{21}\frac{u^1}{u^2} & -(\frac{u^1}{u^2}+\tau)(m_{11} + m_{21}\frac{u^1}{u^2}) \\ m_{21} & -(\frac{u^1}{u^2}+\tau)m_{21} \end{pmatrix},$$
$$H(U) = e^{2m\omega}\begin{pmatrix} n_1u^2 + n_2u^1 \\ n_2u^2 \end{pmatrix}, \tag{80}$$

and $Q_0 = I + \tau u^2\partial_{u^1}$, $Q_1 = 0$, $Q_2 = -\frac{1}{m}u^2\partial_{u^1}$, $\omega = \frac{u^1}{u^2} - \tau\ln u^2$, $m \neq 0$;

$$F(U) = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad G(U) = \begin{pmatrix} m_{11}\frac{u^1}{u^2} & -m_{11}(\frac{u^1}{u^2})^2 \\ m_{21} & -m_{21}\frac{u^1}{u^2} \end{pmatrix}, \qquad H(U) = \left(\frac{u^1}{u^2}\right)^2\begin{pmatrix} n_1u^1 \\ n_2u^2 \end{pmatrix}, \tag{81}$$

and $Q_0 = I$, $Q_1 = 0$, $Q_2 = u^2\partial_{u^2}$;

$$F(U) = -\frac{1}{2}\begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix}, \qquad G(U) = e^{-\frac{1}{2\triangle}\omega}\begin{pmatrix} 2k_1\vec{k}\vec{m} - \frac{2u^1}{\vec{u}^2}\vec{m}\vec{u} & 2k_1\vec{k}^{\perp}\vec{m} - \frac{2u^2}{\vec{u}^2}\vec{m}\vec{u} \\ 2k_2\vec{k}\vec{m} - \frac{2u^2}{\vec{u}^2}\vec{m}\vec{u} & 2k_2\vec{k}^{\perp}\vec{m} + \frac{2u^1}{\vec{u}^2}\vec{m}\vec{u} \end{pmatrix},$$
$$H(U) = e^{-\frac{1}{\triangle}\omega}\begin{pmatrix} \vec{n}\vec{u}\cos\frac{m_3}{\triangle}\omega + \vec{n}\vec{u}^{\perp}\sin\frac{m_3}{\triangle}\omega \\ -\vec{n}\vec{u}^{\perp}\cos\frac{m_3}{\triangle}\omega + \vec{n}\vec{u}\sin\frac{m_3}{\triangle}\omega \end{pmatrix}, \tag{82}$$

and $Q_0 = k_1I - k_2J$, $Q_1 = 0$, $Q_2 = k_3I + m_3J$, $\omega = k_2\ln\vec{u}^2 + 2k_1 \operatorname{arctg}\frac{u^2}{u^1}$, $\triangle = k_2k_3 - k_1m_3$, $\vec{m} = (m_1, m_2)$.

Proof. We consider nonlinearities (54). Extension of the Galilean algebra by mass operator is possible only at $\tau = 0$. According to Table 4, the operator of scale transformations has the form $D = 2x_0\partial_0 + x_1\partial_1 + k_3\partial_{u^1} - u^2\partial_{u^2}$. Since algebra (36) includes operator $Q_0 = \partial_{u^1}$ under the condition, that nonlinearities of system (1) has the form (54) thus, without loss of generality, we can assume that $k_3 = 0$. Therefore its coordinates are

$$\xi^0 = 2x_0, \qquad \xi^1 = x_1, \qquad \eta^a = -\delta_{a2}u^2. \tag{83}$$

Substituting (83) into (16)–(20), we obtain

$$\delta_{a2}f^{2b} - \delta_{b2}f^{a2} = 0, \tag{84}$$
$$-\delta_{c2}u^2g^{ab}_{u^c} + g^{ab} - \delta_{b2}g^{a2} + \delta_{a2}g^{2b} = 0, \tag{85}$$
$$-u^2h^a_{u^2} + 2h^a + \delta_{a2}h^2 = 0. \tag{86}$$

Substituting (54) into the system of equations (84)–(86) and solving it, we get that $\lambda_{12} = \lambda_{21} = m_{11} = m_{21} = m_{22} = n_1 = n_2 = 0$. Thus, nonlinearities F, G and H have the form (75).

The first item of the theorem is proved.

Using similar reasoning, we can prove other items of this theorem. □

5.6. Invariance of system (1) under the generalized Galilean algebra with mass operator

We study which functional matrices F, G and H allow system (1) to be invariant under the algebra $AG_2^M(1,1)$.

The following statement is true.

Theorem 7. *The system of equations (1) is invariant under the generalized Galilean algebra with mass operator if and only if it has the following form, to within equivalence transformations (23), (24):*

$$U_0 = \partial_1\left[\begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix} U_1\right] + \begin{pmatrix} -u^2 & m_{12} \\ 0 & -u^2 \end{pmatrix} U_1 + \frac{(u^2)^2}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{87}$$

and $Q_0 = \partial_{u^1}$, $Q_1 = \partial_{u^2}$, $Q_2 = (\lambda_{11} + m_{12})\partial_{u^1} - u^2\partial_{u^2}$, $Q_3 = 0$;

$$U_0 = \partial_1\left[\begin{pmatrix} \lambda_{11} & -\frac{u^1}{u^2} \\ 0 & -\frac{1}{2} \end{pmatrix} U_1\right] + \begin{pmatrix} m_{11}u^1 & 0 \\ m_{21}u^2 & 0 \end{pmatrix} U_1 + \left(u^1\right)^2\begin{pmatrix} n_1u^1 \\ n_2u^2 \end{pmatrix}, \tag{88}$$

and $Q_0 = u^2\partial_{u^2}$, $Q_1 = 0$, $Q_2 = -u^1\partial_{u^1} + \frac{1}{2}u^2\partial_{u^2}$, $Q_3 = 0$;

$$U_0 = \partial_1\left[\begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix} U_1\right] + \begin{pmatrix} -u^1 & 0 \\ m_{21}u^2 & -(2\lambda_{22}+1)u^1 \end{pmatrix} U_1 + \left(u^1\right)^2u^2\begin{pmatrix} 0 \\ \lambda_{22} + \frac{1}{2} \end{pmatrix}, \tag{89}$$

and $Q_0 = u^2\partial_{u^2}$, $Q_1 = \partial_{u^1}$, $Q_2 = -u^1\partial_{u^1} + (\lambda_{22} + m_{21})u^2\partial_{u^2}$, $Q_3 = 0$;

$$U_0 = \partial_1\left[\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2s} \end{pmatrix} U_1\right] + \omega^2\begin{pmatrix} -sm_{11} & m_{11}\frac{u^1}{u^2} \\ -sm_{12}\frac{u^2}{u^1} & m_{12} \end{pmatrix} U_1 + \omega^4\begin{pmatrix} n_1u^1 \\ n_2u^2 \end{pmatrix}, \tag{90}$$

and $Q_0 = u^1\partial_{u^1} + su^2\partial_{u^2}$, $Q_1 = 0$, $Q_2 = -\frac{1}{2}(I + su^2\partial_{u^2})$, $Q_3 = 0$, $\omega = \frac{u^2}{(u^1)^s}$, $s \neq 0$;

$$U_0 = \partial_1\left[\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} U_1\right] + \frac{u^2}{u^1}\begin{pmatrix} n_1u^1 \\ n_2u^2 \end{pmatrix}, \tag{91}$$

and $Q_0 = I$, $Q_1 = 0$, $Q_2 = (-\frac{1}{2} + \frac{n_1}{n_1-n_2})I - 2u^2\partial_{u^2}$, $Q_3 = \frac{1}{n_1-n_2}u^1\partial_{u^2}$, $n_1 \neq n_2$;

$$\begin{aligned} U_0 = {} & \partial_1\left[-\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} U_1\right] + e^{-2\omega}\begin{pmatrix} m_{11} + m_{21}\frac{u^1}{u^2} & -(\frac{u^1}{u^2}+1)(m_{11} + m_{21}\frac{u^1}{u^2}) \\ m_{21} & -(\frac{u^1}{u^2}+1)m_{21} \end{pmatrix} U_1 \\ & + e^{-4\omega}\begin{pmatrix} n_1u^2 + n_2u^1 \\ n_2u^2 \end{pmatrix}, \end{aligned} \tag{92}$$

and $Q_0 = I + u^2\partial_{u^1}$, $Q_1 = 0$, $Q_2 = -\frac{1}{2}I$, $Q_3 = 0$, $\omega = \frac{u^1}{u^2} - \ln u^2$;

$$\begin{aligned} U_0 = {} & \partial_1\left[-\frac{1}{2}\begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix} U_1\right] + e^{\frac{1}{k_2}\omega}\begin{pmatrix} 2k_1\vec{k}\vec{m} - \frac{2u^1}{\vec{u}^2}\vec{m}\vec{u} & 2k_1\vec{k}^{\perp}\vec{m} - \frac{2u^2}{\vec{u}^2}\vec{m}\vec{u} \\ 2k_2\vec{k}\vec{m} - \frac{2u^2}{\vec{u}^2}\vec{m}\vec{u} & 2k_2\vec{k}^{\perp}\vec{m} + \frac{2u^1}{\vec{u}^2}\vec{m}\vec{u} \end{pmatrix} U_1 \\ & + e^{\frac{2}{k_2}\omega}\begin{pmatrix} \vec{n}\vec{u} \\ -\vec{n}\vec{u}^{\perp} \end{pmatrix}, \end{aligned} \tag{93}$$

and $Q_0 = k_1I - k_2J$, $Q_1 = 0$, $Q_2 = -\frac{1}{2}I$, $Q_3 = 0$, $\vec{m} = (m_1, m_2)$, $\omega = k_2\ln\vec{u}^2 + 2k_1\operatorname{arctg}\frac{u^2}{u^1}$, $|\vec{k}| = 1$.

Proof. We consider the system (1) with nonlinearities (75). For the system (1) with such nonlinearities the generalization of the extended Galilean algebra with mass operator, according to Table 5, is possible only at $Q_3 = 0$. Then operator Π has the following coordinates:

$$\xi^0 = x_0^2, \qquad \xi^1 = x_0 x_1, \qquad \eta^a = \delta_{a1}\left(\frac{x_1^2}{2} + k_3 x_0\right) + \delta_{a2}\left(-x_0 u^2 + x_1\right). \tag{94}$$

We substitute (94) into (16)–(20) and split the obtained equations into x_0 and x_1. As a result, we obtain system of equations (70) and (71), performed according to Theorem 5 and (84)–(86), which satisfy the previous theorem and the following system:

$$f^{a1} + g^{a2} + \delta_{a2} u^2 - \delta_{a1} k_3 = 0. \tag{95}$$

From (75), (95), it follows that $k_3 = \lambda_{11} + m_{12}$. Thus, system (1) with nonlinearities (75) has the form (87).

So, the first item of the theorem is proved. The other items of this theorem are proved in the same way.

The theorem is proved. □

6. Exact solutions of the Van der Waals system

We consider the system, which has been obtained in the second case of Theorem 4. At $\lambda_{12} = \lambda_{21} = \nu_1 = \nu_2 = \mu_{22} = 0$ it has the form

$$\begin{aligned} u_0^1 + u^1 u_1^1 &= \lambda_{11} u_{11}^1 + \mu_{12} u^2 u_1^2, \\ u_0^2 + u^1 u_1^2 &= \lambda_{22} u_{11}^2 - u^2 u_1^1. \end{aligned} \tag{96}$$

System (96) is a system of equations of the Van der Waals fluid, where u^1 is the velocity of the fluid, u^2 is the density, λ_{11} is the kinematic viscosity coefficient, λ_{22} is the diffusion coefficient, μ_{12} is the convection coefficient. This system is invariant under the generalized Galilean algebra

$$AG_2(1,1) = \langle \partial_0,\ \partial_1,\ G = x_0\partial_1 + \partial_{u^1},\ D = 2x_0\partial_0 + x_1\partial_1 - I,\ \Pi = x_0^2\partial_0 + x_0x_1\partial_1 + x_1\partial_{u^1} - x_0 I \rangle \tag{97}$$

We use the symmetry properties of the system (96) for finding its exact solutions.

The nonequivalent ansätze, which were constructed using the operators of algebra (97), have the form

(1) $u^a = \varphi^a(\omega)$, $\omega = x_1 - kx_0$;
(2) $u^1 = \varphi^1(\omega) + kx_0$, $u^2 = \varphi^2(\omega)$, $\omega = x_1 - \frac{k}{2}{x_0}^2$;
(3) $u^a = \frac{\varphi^a(\omega)}{\sqrt{x_0}}$, $\omega = \frac{x_1}{\sqrt{x_0}}$;
(4) $u^1 = \frac{\varphi^1(\omega) + x_0\omega}{\sqrt{x_0^2+1}}$, $u^2 = \frac{\varphi^2(\omega)}{\sqrt{x_0^2+1}}$, $\omega = \frac{x_1}{\sqrt{x_0^2+1}}$,

where $\varphi^a(\omega)$ – arbitrary smooth functions, k – an arbitrary constant.

These ansätze reduce system (96) to the following systems of ODEs

(1) $\lambda_{11}(\varphi^1)'' + (k - \varphi^1)(\varphi^1)' + \mu_{12}\varphi^2(\varphi^2)' = 0,$
$\lambda_{22}(\varphi^2)'' + (k - \varphi^1)(\varphi^2)' - \varphi^2(\varphi^1)' = 0;$
(2) $\lambda_{11}(\varphi^1)'' - \varphi^1(\varphi^1)' + \mu_{12}\varphi^2(\varphi^2)' - k = 0,$
$\lambda_{22}(\varphi^2)'' - (\varphi^1\varphi^2)' = 0;$
(3) $2\lambda_{11}(\varphi^1)'' + (1 - 2(\varphi^1)')\varphi^1 + \omega(\varphi^1)' + 2\mu_{12}\varphi^2(\varphi^2)' = 0,$
$2\lambda_{22}(\varphi^2)'' + (\omega - 2\varphi^1)(\varphi^2)' + (1 - 2(\varphi^1)')\varphi^2 = 0;$

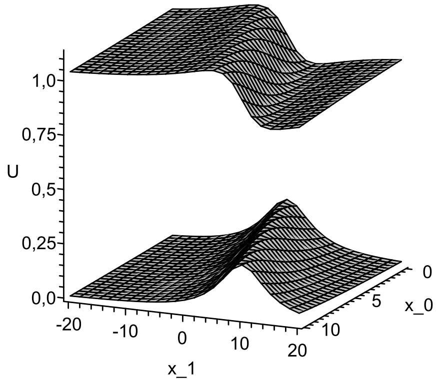

Fig. 1. Exact solution (99) at $C = 6$, $\lambda_{11} = 0.6$, $\mu_{12} = 1$, $k = 1$.

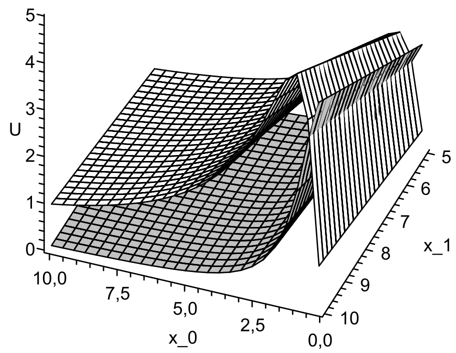

Fig. 2. Exact solution (100) at $\lambda_{11} = 0.2$, $\mu_{12} = 5$, $\lambda_{22} = 0.5$.

(4) $\lambda_{11}(\varphi^1)'' - \varphi^1(\varphi^1)' + \mu_{12}\varphi^2(\varphi^2)' - \omega = 0,$
$\lambda_{22}(\varphi^2)'' - (\varphi^1\varphi^2)' = 0.$

Solving the first reduced system and using a corresponding ansatz, we find the solutions of system (96) at $\lambda_{22} = \lambda_{11}$

$$u^1 = \frac{2\alpha\lambda_{11}(Ce^{\alpha\omega} + 1)}{(Ce^{\alpha\omega} + 1)^2 + 4\mu_{12}e^{2\alpha\omega}} + k - \alpha\lambda_{11}, \qquad u^2 = \frac{4\alpha\lambda_{11}e^{\alpha\omega}}{(Ce^{\alpha\omega} + 1)^2 + 4\mu_{12}e^{2\alpha\omega}}; \tag{98}$$

$$u^1 = k - \frac{2\lambda_{11}\omega}{\omega^2 + \beta}, \qquad u^2 = \frac{C}{\omega^2 + \beta}, \tag{99}$$

where $\omega = x_1 - kx_0$, $\beta = \frac{C^2\mu_{12}}{4\lambda_{11}^2}$; α, k, C – const. Solution (99) at $C = 6$, $\lambda_{11} = 0.6$, $\mu_{12} = 1$, $k = 1$ is presented in Fig. 1.

Solving the forth reduced system, we find the solution of system (96):

$$u^1 = \frac{x_0x_1}{x_0^2 + 1} - \frac{\lambda_{22}}{x_1}, \qquad u^2 = \frac{1}{\sqrt{\mu_{12}}}\left(\frac{x_1}{x_0^2 + 1} + \frac{\sqrt{\lambda_{22}(\lambda_{22} - 2\lambda_{11})}}{x_1}\right) \tag{100}$$

Solution (100) at $\lambda_{11} = 0.2$, $\mu_{12} = 5$, $\lambda_{22} = 0.5$ is presented in Fig. 2.

In Figs. 1 and 2 the upper graphs are u^1, and lower graphs are u^2. The solutions, which are shown in the figures, are continuous, bounded and non-negative. So we may suggest that they can be used for description of some specific physical processes.

Solutions (98), (99) are plane wave solutions, but if they are reproduced by transformations, which are generated by the projective operator Π:

$$x_0' = \frac{x_0}{1-\theta x_0}, \qquad x_1' = \frac{x_1}{1-\theta x_0}, \qquad (u^1)' = \frac{u^1 - \theta x_1}{1-\theta x_0}, \qquad (u^2)' = \frac{u^2}{1-\theta x_0},$$

where θ – group parameter, then we obtain the solutions of system (96), which are non-plane wave solutions,

$$u^1 = \frac{k - x_1\theta}{1-\theta x_0} - \frac{2\lambda_{11}(x_1 - kx_0)}{(x_1-kx_0)^2 + C^2\mu_{12}(1-\theta x_0)^2}, \qquad u^2 = \frac{2C\lambda_{11}(1-\theta x_0)}{(x_1-kx_0)^2 + C^2\mu_{12}(1-\theta x_0)^2},$$
$$u^1 = \frac{1}{1-x_0\theta}\left(\frac{2\alpha\lambda_{11}(Ce^{\alpha\omega}+1)}{(Ce^{\alpha\omega}+1)^2 + 4\mu_{12}e^{2\alpha\omega}} - x_1\theta + k - \alpha\lambda_{11}\right), \qquad u^2 = \frac{4\alpha\lambda_{11}e^{\alpha\omega}}{(1-x_0\theta)((Ce^{\alpha\omega}+1)^2 + 4\mu_{12}e^{2\alpha\omega})},$$

where $\omega = \frac{x_1 - kx_0}{1-\theta x_0}$.

7. Conclusions

After analyzing the results, we can see that some of the obtained systems are a generalization of the previously known ones. System (96) is a system of equations of the Van der Waals fluid that is effectively used for description of processes of the kinetic molecular theory of gases and liquids [23]. System (88) summarizes a system of chemotaxis equations [47], which describes the formation and spread of the Adler chemotactic rings and various processes of the structure formation in bacterial colonies in their interaction. System (90) is a nonlinear system of convection–diffusion equations, obtained in papers [9] and [48]. Systems (89), (91) and (92), (93) generalize some results, which were obtained in works [4–7] and [33–38] respectively, where the group classification of reaction–diffusion systems has been studied. If in system (93) we proceed to a function of a complex variable, we obtain a generalization of the Ginzburg–Landau equation, which is the main nonlinear equation of physics of non-equilibrium environment and describes a diffuse chaos and dissipative structures in hydrodynamics, physics of lasers and chemical kinetics (see, e.g., [11,19,26,32])

$$\psi_0 = -\frac{k}{2}\psi_{11} + \left[\frac{m}{2}\left(2k_1 k\psi^*\psi_1 - \left(|\psi|^2\right)_1\right) + n|\psi|^4 e^{2w}\right]e^{2w}\psi, \tag{101}$$

where $\psi = u^1 + iu^2$, $k, m, n \in \mathbb{C}$. Symmetry properties of the Ginzburg–Landau equation have been studied in papers [2,34].

At $k_1 = 0$, from Eq. (101) we can get a generalization of the Schrödinger equation with a derivative nonlinearity (see, e.g., [3,13–16,20,21,43])

$$i\psi_0 = \frac{1}{2}\psi_{11} + \left[\alpha\left(|\psi|^2\right)_1 + \beta|\psi|^4\right]\psi \tag{102}$$

where $\alpha, \beta \in \mathbb{C}$ that is invariant under the generalized Galilean algebra with mass operator $AG_2^M(1,1)$.

Eq. (102) belongs to the class of equations

$$i\psi_0 = -\frac{1}{2}\psi_{11} + \left(\lambda_1 + \lambda_2|\psi|^2 + \lambda_3|\psi|^4 + \lambda_4\partial_1|\psi|^2\right)\psi + \left(\lambda_5 + \lambda_6|\psi|^2\right)\partial_1\psi,$$

which is used for modeling of wave processes in different parts of physics, for example, in nonlinear optics. They include, for instance, the Alfvén waves with circular polarization, which are magnetohydrodynamical waves, spreading in plasma with a magnetic field, the Stokes waves in a fluid of finite depth, etc. (see, e.g., [28–31,44]).

Thus, in this paper we have found, to within equivalence transformations (23) and (24) nonlinearities f^{ab}, g^{ab}, h^a, which allow system (1) to be invariant under the Galilean algebra (with and without mass operator) and their main extensions by operators of scale and projective transformations. Among the obtained systems there are generalizations of such well-known equations as the Schrödinger equation, the Ginzburg–Landau equation, a chemotaxis system, the Van der Waals system, etc. Since the obtained systems have certain symmetry properties, which are a certain characteristic of similar systems, describing processes, based on the principle of Galilean relativity, they are more suitable for a description of such processes.

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