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ADAPTIVE CONTROL

Stabilization of a Nonlinear Multivariable Discrete-Time Time-Invariant Plant with Uncertainty on a Linear Pseudoinverse Model

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Abstract—This paper addresses the problem of the robust stabilization of a nonlinear multivariable time-invariant plant on a semi-infinite discrete time interval under arbitrary nonmeasurable bounded additive disturbances. A guaranteed value of the quality criterion is given by the functional representing the weighted sum of the limiting norms of the control vectors and output variables. To generate control actions, a controller containing a linear generalized inverse model and discrete-time integrators is introduced into the feedback loop. Sufficient conditions for the robust stability of the control system, as well as the conditions for the ultimate boundedness of all its signals (dissipativity conditions), are formulated. A corresponding simulation example is presented.

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INTRODUCTION

The so-called inverse operator method, which was heuristically proposed in the pioneering works of domestic researchers in the mid-1960s [2, 3], is known [1] to be an effective tool for improving the performance of control systems over multivariable time-invariant plants under complete and incomplete information about the characteristics of the plants and the uncontrollable disturbances acting on them. This method is implemented by a controller based on an inverse model of a plant. It was found [1] that this method can formally be reduced to solving classical inverse problems of dynamics [4]. The recent results in this direction were presented in [5], where the optimal control systems based on a generalized inverse (pseudoinverse) model of a linear multivariable time-invariant plant were considered given an arbitrary matrix of gains for direct connections and interconnections on the assumption that the system designer a priori knows the values of these coefficients.

Unfortunately, when solving a wide class of real-life control problems, some control decisions have to be made under conditions of a priori uncertainty about the mathematical model of a plant. To deal with this uncertainty, the robust control theory was developed. Robust control is known [6–8] to provide a guaranteed value of a certain functional characterizing the performance of a control system over any plant of a given family [9]. The fundamental results obtained in this field during the last two decades were summarized in [7–10]. These monographs deal mostly with robust stability and robust stabilizability of continuous- and discrete-time linear systems.

Recently, many researchers have turned their attention to the robust control of uncertain nonlinear plants. Particularly, in [11, Chapt. 10], some general robust control problems were solved for nonlinear multivariable continuous-time plants. In [12], a new method was proposed for the steady-state robust control of nonlinear continuous-time dynamic plants under conditions of uncertainty by using their robust linear models. It is worth noting that the problem of stabilization of a nonlinear multivariable time-invariant plant on a semi-infinite discrete time interval by using its fixed linear model under insufficient a priori information about its steady-state characteristics was addressed in the early 1970s [13]. The results of this work made a conceptual contribution to the development of a sufficiently general approach to the robust stabilization of uncertain nonlinear multivariable time-invariant plants with linear feedbacks that

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was presented by one of the authors of this paper at the 19th IFAC World Congress [14]. This approach is based on a concept whereby the matrix of a reference linear model of a nonlinear multivariable plant undergoes pseudoinversion just as in the linear case, where the number of control inputs does not exceed the number of output variables [5].

This paper extends the approach proposed in [14] by formulating new, sufficient conditions for the robust stability of control systems over a certain class of nonlinear multivariable time-invariant plants with fixed linear models. In contrast to other well-known solutions, these conditions are reduced to obtaining the upper estimate for the norm of an interval matrix; this seems to be a far more adequate approach than finding the upper estimate for the norm of a matrix characterizing the relative deviations of differential gains from the corresponding elements of a matrix of a reference linear model over the whole unbounded set of control vectors (as in [13, p. 17]).

Speaking of practical use, the results obtained in this work are intended mostly for designing multivariable control systems of continuous technological processes such as those described in [15, 16].

1. FORMULATION OF THE PROBLEM

Suppose that we have a nonlinear multivariable time-invariant plant operating on a semi-infinite discrete time interval $n = 0, 1, 2, \ldots$, with an m-dimensional vector $y_n = [y_n^{(1)}, \ldots, y_n^{(m)}]^{\mathrm{T}}$ of output variables measured at each nth time step and an r-dimensional vector $u_n = [u_n^{(1)}, \ldots, u_n^{(r)}]^{\mathrm{T}}$ of controls (hereinafter, T denotes transposition). The output $y_n \in \mathrm{R}^m$ and the control $u_n \in \mathrm{R}^r$ are assumed to be related as follows:

$$y_n = \varphi(u_n) + v_n, \quad n \in N_+. \tag{1.1}$$

Here, $\varphi\colon \mathrm{R}^r \to \mathrm{R}^m$ is a certain a priori unknown nonlinear vector function

$$\varphi(u) = [\varphi^{(1)}(u), \ldots, \varphi^{(m)}(u)]^{\mathrm{T}}, \tag{1.2}$$

$v_n = [v_n^{(1)}, \ldots, v_n^{(m)}]^{\mathrm{T}}$ is the m-dimensional vector of additive uncontrollable disturbances (inferences), and $N_+ := \{0, 1, 2, \ldots\}$ is the infinite ordered set of nonnegative integers. Consider the case where the number r of input variables $u_n^{(1)}, \ldots, u_n^{(r)}$ does not exceed the number m of output variables $y_n^{(1)}, \ldots, y_n^{(m)}$:

$$r \le m. \tag{1.3}$$

Following [13], we assume that each ith component of $\varphi^{(i)}(u)$, $i = \overline{1, m}$, in (1.2) is a continuously differentiable function of $u^{(1)}, \ldots, u^{(r)}$. Then, a significant assumption can be made that all r partial derivatives of $\partial\varphi^{(i)}(u)/\partial u^{(j)}$ remain uniformly separated from zero and bounded in R^m, while preserving their signs for all u from R^r. This means that

$$\begin{gathered} \underline{b}^{(ij)} \le \partial\varphi^{(i)}(u)/\partial u^{(j)} \le \overline{b}^{(ij)}, \quad 0 < \underline{b}^{(ij)}\overline{b}^{(ij)} < \infty, \\ i = \overline{1, m}, \quad j = \overline{1, r}. \end{gathered} \tag{1.4}$$

It can be seen that, on assumption (1.4), $\varphi^{(i)}(u)$ are infinite functions and

$$\inf_{u \in \mathrm{R}^r} \varphi^{(i)}(u) = -\infty, \quad \sup_{u \in \mathrm{R}^r} \varphi^{(i)}(u) = +\infty \tag{1.5}$$

for all $i = \overline{1, m}$. All the bounds $\underline{b}^{(ij)}$ and $\overline{b}^{(ij)}$ in (1.4) are assumed to be known a priori.

As in [9], we assumed that, for each ith component $v_n^{(i)}$ of the vector v_n, the sequence $\{v_n^{(i)}\} := \{v_0^{(i)}, v_1^{(i)}, \ldots\}$ gives rise to an infinite sequence of arbitrary modulo-bounded variables [7, 9, 10], i.e.,

$$|v_n^{(i)}| \le \varepsilon^{(i)} < \infty, \quad i = \overline{1, m}, \quad n \in N_+; \tag{1.6}$$

in this case, $\varepsilon^{(i)}$ are not necessarily known. (Using the notation accepted in the modern control theory, constraint (1.4) can be rewritten as

$$\{v_n\} \in \underbrace{\ell_\infty \times \ldots \times \ell_\infty}_{m}, \tag{1.7}$$

where ℓ_∞ denotes the space of all possible sequences of bounded scalar values $x_n \in \mathrm{R}^1$ with the ℓ_∞ norm:

$$\|x\|_\infty = \sup_{n \in N_+} |x_n| < \infty;$$

see [9, p. 29].)

Consider the linear feedback system

$$u_{n+1} = u_n + A e_n, \tag{1.8}$$

$$e_n := y^0 - y_n, \tag{1.9}$$

which is based on a single-step iterative control strategy and stabilizes the output variables $y_n^{(i)}$ on the given levels $y^{0(i)}$ ($y^{0(i)} \equiv \text{const}$ for all $i = \overline{1, m}$); this system was proposed in [13, p. 19] for multivariable time-invariant plants. Here, A is the $r \times m$ constant matrix of interconnections that depends in a certain way on the matrix B_0 of the so-called reference linear model of nonlinearity $\varphi(u)$ [13, p. 25]. (The matrices A and B_0 are selected below.)

Without loss of generality, the components of the vector $y^0 = [y^{0(1)}, \ldots, y^{0(m)}]^{\mathrm{T}}$ are assumed to satisfy the requirement

$$|y^{0(1)}| + \ldots + |y^{0(m)}| \neq 0,$$

which is basically interpreted as $\|y^0\| \neq 0$. This means that $y^{0(i)} \neq 0$ at least for one i.

It is also assumed that, for $r = m$, there certainly is a solution of the vector equation

$$\varphi(u) = y^0,$$

i.e., a solution to the system of nonlinear algebraic equations

$$\varphi^{(i)}(u) = y^{0(i)}, \quad i = \overline{1, r}, \tag{1.10}$$

in the components of the vector $u \in \mathrm{R}^r$. It is condition (1.5) that is necessary for a solution of system (1.10) to exist.

The problem consists in finding, based on the assumptions made above about the nonlinearity $\phi(u)$, the conditions that, first, guarantee the robust stability of the closed-loop system (1.1), (1.8), and (1.9) for the whole family of possible nonlinearities satisfying interval constraints (1.4) at $v_n^{(i)} \equiv 0, i = \overline{1, m}$, and, second, ensure the ultimate boundedness [6] in terms of

$$\overline{\lim_{n\to\infty}}(\|y_n\| + \tau\|u_n\|) < \infty \quad (\tau > 0) \tag{1.11}$$

in the class of all possible sequences $\{v_n\}$ of form (1.7).

2. PRELIMINARIES

With $b^{(ij)}(u) := \partial\varphi^{(i)}(u)/\partial u^{(j)}$, introduce the matrix

$$B(u) = \begin{pmatrix} b^{(11)}(u) & \ldots & b^{(1r)}(u) \\ & \vdots & \\ b^{(m1)}(u) & \ldots & b^{(mr)}(u) \end{pmatrix}, \tag{2.1}$$

which is an $m \times r$ Jacobian whose elements are a kind of dynamic gains from the jth control $u_n^{(j)}$ to the ith output $y_n^{(i)}$ (for all fixed $u_n \in \mathrm{R}^r$). Due to (1.4), the rank of this matrix, according to (1.3) and (2.1), satisfies the condition

$$1 \le \operatorname{rank} B(u) \le r. \tag{2.2}$$

Suppose that a pair of vectors u^e and $y^e = \varphi(u^e)$ defines an equilibrium state $\{u^e, y^e\}$ of the closed-loop system (1.1), (1.8), and (1.9) under zero disturbances. Upon denoting a k-dimensional null vector by

$$0_k := [\underbrace{0, \ldots, 0}_{k}]^{\mathrm{T}},$$

it is easy to see that u^e is a solution of the equation

$$A(y^0 - \varphi(u)) = 0_r, \tag{2.3}$$

which follows from (1.8) taking into account (1.1) and (1.9) at $u_{n+1} = u_n = u^e$.

It is believed [13] that, if a nonlinear system (1.1), (1.8), and (1.9) has an equilibrium state under $v_n \equiv 0_m$, which is defined by (2.3), then the requirement

$$\sup_{u \in \mathrm{R}^r} \|I_r - AB(u)\| < 1 \tag{2.4}$$

for any matrix norm $\|\cdot\|$, where I_r is an $r \times r$ identity matrix, is a sufficient condition for the asymptotic stability of this system under arbitrary initial conditions. This requirement is essentially a condition for the asymptotic stability of the whole control system comprising the nonlinear plant (1.1) with fixed nonlinearity $\varphi(u)$ and the linear feedback (1.8) and (1.9).

Condition (2.4) implies that the stability of the system under consideration cannot be guaranteed if the matrix A or $B(u)$ is rank deficient for at least one $u \in \mathrm{R}^r$, i.e., rank $A < r$ or rank $B(u) < r$. Indeed, in this case, at least one of r eigenvalues $\lambda_i(Q_u(u))$ of the matrix $Q_u(u) := I_r - AB(u)$ becomes 1. This can be verified using some remarkable properties of ranks, eigenvalues, and norms from matrix theory, particularly, the property of the rank of the product of two matrices

$$\operatorname{rank} P_1P_2 \le \min\{\operatorname{rank} P_1, \operatorname{rank} P_2\},$$

which is implied by the Frobenius inequality [17, Part I, Subsect. 2.17.1], the property

$$\lambda_i(\alpha I_r + \beta P) = \alpha + \beta\lambda_i(P), \quad i = \overline{1, r},$$

of any ith eigenvalue $\lambda_i(P)$ of an arbitrary matrix $P \in \mathrm{R}^{r \times r}$ for arbitrary α, β from R [17, Part I, Subsect. 2.15.3], and the well-known relation

$$\|P\| \ge \max_{i=\overline{1, r}} |\lambda_i(P)|$$

for any matrix norm [9, p. 260] that follows from Brauer's theorem [17, Part III, Subsect. 1.6.5] and Brown's theorem [17, Part III, Subsect. 1.5]. Thus, the requirement

$$\operatorname{rank} A = \operatorname{rank} B(u) = r, \quad \forall u \in \mathrm{R}^r, \tag{2.5}$$

which refines (2.2), is necessary for condition (2.4) to hold.

It can be shown that, in the context of assumption (1.4), the requirement rank $B(u) = r$ for all $u \in \mathrm{R}^r$, which appears in (2.5), is met a priori for $m = r$ if

$$\min\{|\underline{b}^{(jj)}|, |\overline{b}^{(jj)}|\} > \sum_{i \ne j} \max\{|\underline{b}^{(ij)}|, |\overline{b}^{(ij)}|\}, \quad \forall j = \overline{1, r}. \tag{2.6}$$

Conditions (2.6), which are easy to test using a priori information about the boundaries of the elements from the matrix $B(u)$, expressed in terms of (1.4), are obtained by directly applying Hadamard's theorem (also known as the Levy–Desplanque theorem [17, p. 192]) to the matrix $B(u)$ of form (2.1) (see also [18, Subsect. 16.27]).

Following [13], define a matrix $D(u)$ characterizing the relative deviations of the elements in the $m \times r$ Jacobian $B(u)$ from the elements in a certain fixed $m \times r$ matrix $B_0 = (b_0^{(ij)})$ of the reference linear model:

$$B_0 D(u) = B(u) - B_0. \tag{2.7}$$

Then, as shown in [13, p. 27], if the matrix A in control law (1.8) is selected by the formula

$$A = (B_0^{\mathrm{T}} B_0)^{-1} B_0^{\mathrm{T}}, \tag{2.8}$$

then the stability of the control system is guaranteed at the maximum possible degree of inadequacy of the nonlinear plant and its linear model, which is expressed by the inequality

$$\max_{u \in \mathrm{R}^r} \|D(u)\| < 1.$$

Formula (2.8) needs one significant refinement associated with a constraint on selecting the matrix B_0; unfortunately, we failed to find the corresponding guidelines in [13]. The point is that the inversion of the product $B_0^{\mathrm{T}} B_0$, which is required to find the matrix A by this formula, is allowed only if the requirement

$$\det\,(B_0^{\mathrm{T}} B_0) \neq 0$$

imposed on the $r \times r$ matrix $B_0^{\mathrm{T}} B_0$ is met. However, under condition (1.3), this is possible only if

$$\mathrm{rank} B_0 = r, \tag{2.9}$$

i.e., if B_0 has full rank (see [18, Subsect. 4.41]).

Note that, for $r < m$ and under condition (2.9), the right-hand side of (2.8) is nothing but a pseudoinverse matrix $B_0^+ = \beta^{(ij)}$, which is defined as [18, Subsect. 6.46]

$$B_0^+ := (B_0^{\mathrm{T}} B_0)^{-1} B_0^{\mathrm{T}}; \tag{2.10}$$

however, for $r = m$, this right-hand side proves to be a common inverse matrix, i.e., $B_0^+ = B_0^{-1}$.

Thus, it is reasonable to take $A = B_0^+$ in control law (1.8), thereby reducing it to

$$u_{n+1} = u_n + B_0^+ e_n; \tag{2.11}$$

in this case, the stability condition (2.4) is given by the relation

$$\sup_{u \in \mathrm{R}^r} \|I_r - B_0^+ B(u)\| < 1, \tag{2.12}$$

which holds if (1.3) and (2.9) are fulfilled.

Now, relation (2.7) can be rewritten as

$$\Delta(u) = B(u) - B_0. \tag{2.13}$$

Here, $\Delta(u) = (\delta^{(ij)}(u)) = B_0 D(u)$ is obviously the matrix characterizing the absolute deviations of the elements in $B(u)$ from the elements in B_0 that was introduced in [14]. To formulate a condition for the stability of the nonlinear control system (1.1), (1.8), and (1.9) in terms of the matrix $\Delta(u)$, we need the following lemma.

Lemma. A matrix B_0^+ of form (2.10) possesses the property

$$B_0^+ B_0 = I_r. \tag{2.14}$$

This lemma can easily be verified by directly substituting (2.10) into the left-hand side of (2.14).

Using (2.13) and taking into account (2.14), condition (2.12) is rewritten as

$$\sup_{u \in \mathrm{R}^r} \|B_0^+ \Delta(u)\| < 1. \tag{2.15}$$

As $B_0^+ = B_0^{-1}$ for $r = m$, condition (2.15) takes the form

$$\sup_{u \in \mathrm{R}^r} \|B_0^{-1} \Delta(u)\| < 1. \tag{2.16}$$

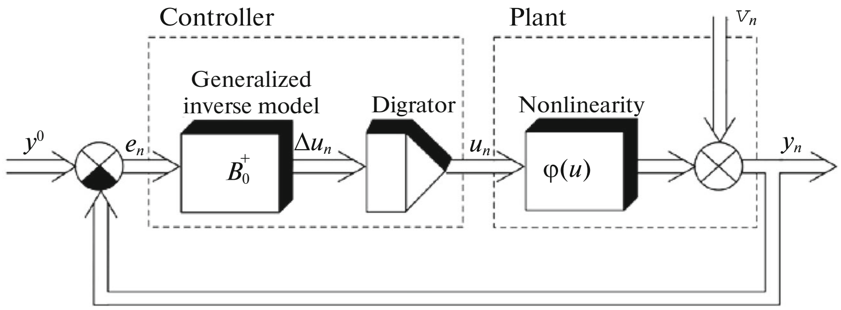

Fig. 1. Configuration of control system based on linear generalized inverse model.

Figure 1 shows the control system of a nonlinear plant that implements the control law (2.11) taking into account (1.9). In this system, the controller is designed as a series connection of a generalized inverse model (described by the transfer matrix B_0^+) and a discrete-time integrator (digrator), which carries out, according to (2.11), the sum accumulation operation

$$u_n = \sum_{i=0}^{n-1} \Delta u_i \quad (u_0 = 0_r),$$

where $\Delta u_n := u_{n+1} - u_n$ is the vector output of the generalized inverse model, which is defined, according to (2.11), as

$$\Delta u_n = B_0^+ e_n$$

(see Fig. 1). The equilibrium state of this system (in which $\Delta u_n \equiv 0_r$) is obviously given by the equation

$$B_0^+(y^0 - \varphi(u)) = 0_r, \tag{2.17}$$

which is obtained by substituting B_0^+ for A in (2.3).

In a one-dimensional case ($r = m = 1$), with the digrator in the feedback loop, under zero disturbances ($v_n^{(1)} \equiv 0$), the linear discrete-time control system always becomes an astatic one, i.e., $\lim_{n\to\infty} e_n = 0$, if the condition for its asymptotic stability is fulfilled.

Remark 1. It is interesting that, in the case of a nonlinear multivariable discrete-time system where $r + m > 2$, the similar limit property $\lim_{n\to\infty} e_n = 0_m$ a priori holds only if $r = m$. This is directly implied by the equilibrium equation (2.17) for $B_0^+ = B_0^{-1}$. Indeed, the kernel of the matrix B_0^{-1}, which appears in this equation, has the dimension $\dim\ker B_0^{-1} = m - \operatorname{rank} B_0^{-1} = m - r = 0$ [17, Subsect. 3.1.5], and the steady-state equilibrium $u^e = u_\infty, y^e = y_\infty$ of this system, in the limiting case, for $n \to \infty$, should satisfy the condition $y^0 - \varphi(u_\infty) \in \ker B_0^{-1}$ due to (2.17). Since, according to (1.9) and taking into account (1.1), for $v_n \equiv 0_m$, the steady-state error is defined as $e_\infty = y^0 - \varphi(u_\infty)$, this certainly implies that $e_\infty = 0_m$. However, if $r < m$, then $\dim\ker B_0^+ = m - \operatorname{rank} B_0^+ = m - r > 0$. In this case, the equilibrium equation (2.17) can theoretically be satisfied even if $e_\infty \neq 0_m$. Thus, for $r < m$, the presence of the digrator in the feedback loop by no means guarantees the astatism of the closed-loop control system (1.1), (1.9), and (2.11).

3. MAIN RESULTS

First, consider a case where $r = m$. Following [9, Subsect. 6.1], select the elements of the matrix B_0 for the reference linear model:

$$b_0^{(ij)} = \frac{\underline{b}^{(ij)} + \overline{b}^{(ij)}}{2}. \tag{3.1}$$

Such a selection gives rise to the r^2-dimensional vector

$$\operatorname{vec} B_0 = [b_0^{(11)}, \ldots, b_0^{(1r)}, \ldots, b_0^{(r1)}, \ldots, b_0^{(rr)}]^{\mathrm{T}},$$

which is formed by stretching the $r \times r$ matrix B_0 into a column (as in [9, p. 220], any vector obtained by the matrix P is denoted by $\operatorname{vec} P$). This vector is practically a Chebyshev center of the bounded uncertainty set

$$\Xi := [\underline{b}^{(11)}, \overline{b}^{(11)}] \times \ldots \times [\underline{b}^{(rr)}, \overline{b}^{(rr)}] \subset \mathrm{R}^{r^2},$$

which, according to (1.4), includes any vector

$$b(u) = [b^{(11)}(u), \ldots, b^{(1r)}(u), \ldots, b^{(r1)}(u), \ldots, b^{(rr)}(u)]^{\mathrm{T}}.$$

Based on (2.13) and taking into account (2.1), for the elements $\delta^{(ij)}(u)$ of the matrix $\Delta(u)$, we have

$$\delta^{(ij)}(u) = b^{(ij)}(u) - b_0^{(ij)}. \tag{3.2}$$

As $b^{(ij)}(u) := \partial\varphi^{(i)}(u)/\partial u^{(j)}$, due to (1.4) and (3.2), we can write

$$\underline{\delta}^{(ij)} \le \delta^{(ij)}(u) \le \overline{\delta}^{(ij)}, \tag{3.3}$$

where

$$\underline{\delta}^{(ij)} = \underline{b}^{(ij)} - b_0^{(ij)}, \qquad \overline{\delta}^{(ij)} = \overline{b}^{(ij)} - b_0^{(ij)}; \tag{3.4}$$

in this case, taking into account (3.1), we have

$$\overline{\delta}^{(ij)} = -\underline{\delta}^{(ij)} = \frac{\overline{b}^{(ij)} - \underline{b}^{(ij)}}{2}. \tag{3.5}$$

Assume that $\det B_0 \ne 0$; otherwise, as shown above, the robust stability of the system under consideration is by no means guaranteed. Under this assumption, the pseudoinverse matrix B_0^+ is known to become a simple inverse matrix B_0^{-1} with a spectral norm

$$\|B_0\|_2 = \max_{i=\overline{1,\, r}} \lambda_i^{1/2}((B_0^{-1})^{\mathrm{T}} B_0^{-1}),$$

where λ_i are the singular values of B_0^{-1} (see [9, p. 259] and [18, p. 103]). Suppose that we have an uncertainty set Ξ such that the relation

$$r \max_{i,j=\overline{1,\, r}} \overline{\delta}^{(ij)} < 1/\|B_0^{-1}\|_2$$

holds; taking into account (3.5), this relation is rewritten as

$$R = r \max_{i,j=\overline{1,\, r}} (\overline{b}^{(ij)} - \underline{b}^{(ij)})/2 < 1/\|B_0^{-1}\|_2. \tag{3.6}$$

Here, R is the radius of a sphere in the space R^{r^2} with its center at the point $\operatorname{vec} B_0$; as shown in Fig. 2, this sphere a priori encloses the set Ξ. Indeed, for the distance from the point $\operatorname{vec} B_0$ to the most distant point of the set Ξ (which equals the semidiameter of this set), the following relations hold:

$$\frac{1}{2}\operatorname{diam} \Xi = \frac{1}{2}\sqrt{\underbrace{(\overline{b}^{(11)} - \underline{b}^{(11)})^2 + \ldots + (\overline{b}^{(rr)} - \underline{b}^{(rr)})^2}_{r^2}} \le r \max_{i,j=\overline{1,\, r}} \frac{\overline{b}^{(ij)} - \underline{b}^{(ij)}}{2}.$$

If relation (3.6) holds, then, due to (3.2)–(3.4), we definitely have

$$\|\Delta(u)\|_2 < r \max_{i,j=\overline{1,\, r}} \overline{\delta}^{(ij)}, \tag{3.7}$$

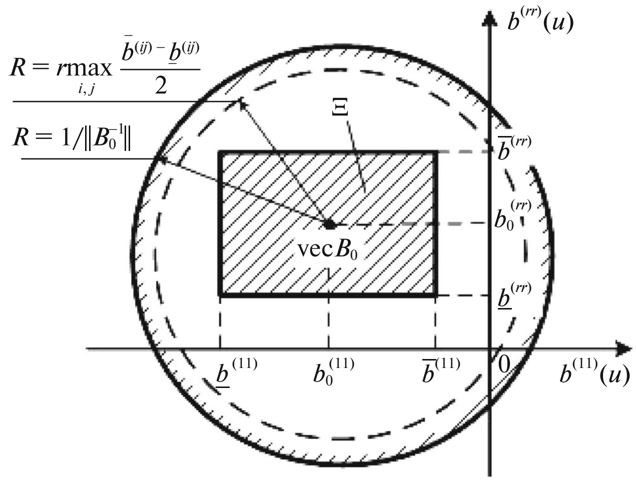

Fig. 2. Geometric interpretation of uncertainty set Ξ and condition (3.6).

because, according to [18, p. 104], the matrix norm $\|\cdot\|_2$ possesses the property

$$\|\Delta(u)\|_2 < r \max_{i,j=\overline{1,r}} |\delta^{(ij)}(u)|$$

for any fixed $u \in \mathrm{R}^r$, and

$$\sup_{u\in \mathrm{R}^r} \max_{i,j=\overline{1,r}} |\delta^{(ij)}(u)| = \max_{i,j=\overline{1,r}} \bar{\delta}^{(ij)}.$$

However, if (3.7) holds, then the set of matrices $B(u)$, for which $b(u) \in \Xi$, is a priori a set of nondegenerate matrices (see [9, Lemma 7.2]). Moreover, in this case,

$$\sup_{u\in \mathrm{R}^r} \|B_0^{-1}\Delta(u)\|_2 < 1, \tag{3.8}$$

because, due to the well-known property of matrix norms (see, for example, [18, p. 103]), we can write

$$\sup_{u\in \mathrm{R}^r} \|B_0^{-1}\Delta(u)\|_2 \le \|B_0^{-1}\|_2 \sup_{u\in \mathrm{R}^r} \|\Delta(u)\|_2,$$

while taking into account the fact that

$$\|B_0^{-1}\|_2 \sup_{u\in \mathrm{R}^r} \|\Delta(u)\|_2 < 1$$

(due to (3.7)). In turn, condition (3.8) duplicates condition (2.16) for the asymptotic stability of system (1.1), (1.9), and (2.11) at $B_0^+ = B_0^{-1}$, which is written in terms of the matrix norm $\|\cdot\|_2$. This implies the following theorem.

Theorem 1. Suppose that $r = m$ and select a matrix B_0 whose elements $b_0^{(ij)}$ are defined by expression (3.1). If $\det B_0 \ne 0$ and condition (3.6) holds, then, under zero disturbances, the closed-loop system (1.1), (1.9), and (2.11) at $B_0^+ = B_0^{-1}$ is robustly stable for the whole family of nonlinearities $\varphi(u)$, that satisfy constraints (1.4).

Unfortunately, condition (3.6) in Theorem 1 sets excessively strict lower and upper bounds for the admissible values of the elements $b^{(ij)}(u)$ in $B(u)$. Moreover, Theorem 1 is valid only for those plants of form (1.1) in which the number of output variables equals the number of control inputs.

In the case where $r \le m$, the problem formulated above is solved as follows. Define an $m \times r$ matrix $B_0 = (b_0^{(ij)})$ whose elements satisfy the condition

$$\underline{b}^{(ij)} < b_0^{(ij)} < \bar{b}^{(ij)}, \quad i = \overline{1, m}, \quad j = \overline{1, r}, \tag{3.9}$$

in such a way that requirement (2.9) is met. (For $r = m$, condition (2.9) means that $\det B_0 \neq 0$.) Suppose that the system (1.1), (1.9), and (2.11) has an equilibrium state given by (2.17), and turn to the sufficient condition (2.15) for the asymptotic stability of this system.

As the relation

$$\sup_{u \in \mathbf{R}^r} \|B_0^+ \Delta(u)\| \leq \max_{\Delta: \delta^{(ij)} \in [\underline{\delta}^{(ij)}, \overline{\delta}^{(ij)}]} \|B_0^+ \Delta\|,$$

where $\Delta = (\delta^{(ij)})$ is a matrix with elements $\delta^{(ij)}$, holds due to (3.2) and (3.3), condition (2.15) is necessarily met if

$$\max_{\Delta: \delta^{(ij)} \in [\underline{\delta}^{(ij)}, \overline{\delta}^{(ij)}]} \|B_0^+ \Delta\|_1 < 1, \tag{3.10}$$

where $\|\cdot\|_1$ is the row norm of the matrix. Now, using the definition

$$\|P\|_1 := \max_{i=\overline{1,\, r}} \sum_{j=1}^{r} |p^{(ij)}|$$

of the row norm of an arbitrary $r \times r$ matrix $P = (p^{(ij)})$ (see [9, p. 259]) and taking into account that each element in any ith column $\delta^{(i)} = [\delta^{(1i)}, \ldots, \delta^{(mi)}]^{\mathrm{T}}$ of the matrix $\Delta = (\delta^{(ij)})$ runs through the values on the interval $\underline{\delta}^{(ji)} \leq \delta^{(ji)} \leq \overline{\delta}^{(ji)}$ no matter what values are taken by the elements in the other $r - 1$ columns, condition (3.10) can be rewritten as

$$q < 1, \tag{3.11}$$

where

$$q := \max_{k=\overline{1,\, r}} \sum_{i=1}^{r} \max_{\underline{\delta}^{(ij)} < \delta^{(ij)} < \overline{\delta}^{(ij)}} \left| \sum_{j=1}^{m} \beta_0^{(kj)} \delta^{(ji)} \right|. \tag{3.12}$$

Hence, the following theorem holds.

Theorem 2. Fix an arbitrary full rank matrix B_0. Suppose that the basic assumptions (1.4) about nonlinearity $\varphi(u)$ hold. Then, under zero disturbances, a closed-loop control system comprising a nonlinear time-invariant plant (1.1) and a linear controller (1.9) and (2.11) is robustly stable for the whole family of nonlinearities that satisfy the interval constraints (1.4) if this system has an equilibrium state and condition (3.11) is fulfilled, in which q is given by (3.12) and $\underline{\delta}^{(ij)}$, $\overline{\delta}^{(ij)}$ are given by (3.4).

Remark 1. The constraints $\underline{b}^{(ij)} \overline{b}^{(ij)} > 0$, $i = \overline{1, m}$, $j = \overline{1, r}$, which appear in (1.4), are used to avoid the case where inevitably $q \geq 1$, i.e., the asymptotic stability condition is not met a priori. The need for these constraints can be explained by rewriting original inequality (3.10) as

$$\max_{B:\ \mathrm{vec}\, B \in \Xi} \|I_r - B_0^+ B\|_1 < 1,$$

which is done using the definition of the uncertainty set

$$\Xi := [\underline{b}^{(11)}, \overline{b}^{(11)}] \times \ldots \times [\underline{b}^{(mr)}, \overline{b}^{(mr)}] \subset \mathrm{R}^{mr},$$

which, obviously, should include the vector $\mathrm{vec}\, B$, where $B := B_0 + \Delta$. It can be seen that the last inequality does not hold if $\underline{b}^{(ij)} \overline{b}^{(ij)} \leq 0$ for all $i = \overline{1, m}$, $j = \overline{1, r}$, because, in this case, the set Ξ certainly contains a null vector 0_{mr} and, therefore, the set of matrices B always contains a zero matrix $0_{m,\, r}$ (here, as in [17, p. 9], and a $m \times r$ zero matrix is denoted by $0_{m,\, r}$).

It turns out that, for $r = m$, the constraints mentioned above can be significantly weakened by substituting them with $\underline{b}^{(ij)} \overline{b}^{(ij)} > 0$, $i, j = \overline{1, m}$. Indeed, these constraints alone guarantee that $\Xi \not\ni 0_{m,m}$.

Remark 3. In contrast to the general sufficient condition (2.15) for stability of the closed-loop system (1.1), (1.9), and (2.11), which seems to be inadequate for practical use (to test this condition formally requires searching through all possible vectors u from R^r), conditions (3.11) and (3.12) of Theorem 2 can be tested by searching through a finite number of values appearing on the right-hand side of (3.12) in the nodes of a polytope of the constraints formed by inequalities (3.3). Meanwhile, by themselves, these conditions limit the admissible deviations of $B(u)$ from B_0 more strictly than condition (2.15). Indeed, the maximization operation on the right-hand side of (3.12) assumes that each element $\delta^{(ij)}$ of the matrix Δ can take any values within a certain range no matter what values are taken by the other r^2-1 elements of this matrix; in contrast, the elements $\delta^{(ij)}(u)$ of the matrix $\Delta(u)$, which appears in (2.15), can take only particular values (depending on $u \in R^r$) within the same range $[\underline{\delta}^{(ij)}, \overline{\delta}^{(ij)}]$.

To facilitate the test of the robustness condition (3.11) and (3.12), an approach similar to that proposed in [9, Subsect. 4.5] for a superstable control system can be used, reducing this test to solving a series of fairly simple linear programming problems. For this purpose, it is sufficient to introduce (see [9, Theorem 4.15]), the auxiliary variables

$$\sigma^{(ki)} = \sum_{j=1}^{m} \beta_0^{(kj)} \delta^{(ji)}, \quad k, i = \overline{1, r},$$

which are linear forms with respect to the vectors $\delta^{(i)} = [\delta^{(1i)}, \ldots, \delta^{(mi)}]^{\mathrm{T}}$ that allow us to transfer from checking inequality (3.11) to solving some pairs of linear programming problems

$$\min \sigma^{(ki)}, \qquad \max \sigma^{(ki)} \tag{3.13}$$

under the constraints

$$\underline{\delta}^{(ij)} \le \delta^{(ij)} \le \overline{\delta}^{(ij)}. \tag{3.14}$$

Eventually, this approach leads to the following corollary of Theorem 2.

Corollary. Suppose that the conditions of Theorem 2 are fulfilled. If

$$\sum_{i=1}^{r} \max\{|\min \sigma^{(ki)}|, |\max \sigma^{(ki)}|\} < 1 \quad \forall\ k = \overline{1, r}, \tag{3.15}$$

where $\min \sigma^{(ki)}$ and $\max \sigma^{(ki)}$ are the solutions of linear programming problems (3.13) and (3.14), then a control system of a nonlinear plant (1.1) with a linear feedback (2.23) is robustly stable.

Thus, when requirement (3.15) is met and $v_n \equiv 0_m$, for any initial $u_0 \in \mathrm{R}^r$, it is guaranteed that system (1.1), (1.9), and (2.11) tends to the equilibrium $\{u^e, y^e\}$ with an unlimited increase in n, i.e.,

$$\lim_{n\to\infty} y_n = y^e, \tag{3.16}$$

where the vector $y^e = \varphi(u^e)$ satisfies (2.17). Of course, for $v_n \not\equiv 0_m$, this tendency does not possess the limit property (3.16). Nevertheless, the dissipativity (ultimate boundedness) of this tendency is guaranteed. This is proved in the following theorem.

Theorem 3 (on dissipativity)**.** Under the conditions of Theorem 2 and on assumption (1.6), the nonlinear closed-loop control system (1.1), (1.9), and (2.11) remains ultimately bounded for $n \to \infty$ if condition (3.11) is fulfilled; in this case,

$$\overline{\lim_{n\to\infty}} \|u_n - u^e\|_\infty \le \|B_0^+\|_1 \varepsilon (1-q)^{-1} < \infty, \tag{3.17}$$

$$\overline{\lim_{n\to\infty}} \|y_n - y^e\|_\infty \le \max_{B:\, \underline{b}^{(ij)} \le b^{(ij)} \le \overline{b}^{(ij)}} \|B\|_1 \|B_0^+\|_1 \varepsilon (1-q)^{-1} + \varepsilon < \infty, \tag{3.18}$$

where $\varepsilon = \max\limits_{i=\overline{1,r}} \varepsilon^{(i)}$.

The proof is conducted in the Appendix.

In terms of the modern control theory, properties (3.17) and (3.18) mean that

$$\{u_n\} \in \underbrace{\ell_\infty \times \ldots \times \ell_\infty}_{r}, \quad \{y_n\} \in \underbrace{\ell_\infty \times \ldots \times \ell_\infty}_{m},$$

which directly implies that relation (1.11) holds. Therefore, the boundedness properties of the sequences $\{u_n\}$ and $\{y_n\}$ (see Theorem 3) eventually provide a solution to the problem formulated in this paper.

Simulation example. Suppose that we have a nonlinear time-invariant plant with two control inputs $u^{(1)}$, $u^{(2)}$ and two outputs $y^{(1)}$, $y^{(2)}$ ($r = m = 2$) that is described by Eq. (1.1) in which the components of the vector function (1.2) are defined as

$$\varphi^{(1)}(u) = \varphi^{(11)}(u^{(1)}) + \varphi^{(12)}(u^{(2)}), \quad \varphi^{(2)}(u) = \varphi^{(21)}(u^{(1)}) + \varphi^{(22)}(u^{(2)}), \tag{3.19}$$

where

$$\begin{aligned} \varphi^{(11)}(u^{(1)}) &= \frac{\alpha_{11}u^{(1)} + \gamma_{11}(u^{(1)})^3}{\eta_{11} + (u^{(1)})^2}, \quad \varphi^{(12)}(u^{(2)}) = -\frac{\alpha_{12}u^{(2)} + \gamma_{12}(u^{(2)})^3}{\eta_{12} + (u^{(2)})^2}, \\ \varphi^{(21)}(u^{(1)}) &= \frac{\alpha_{21}u^{(1)} + \gamma_{21}(u^{(1)})^3}{\eta_{21} + (u^{(1)})^2}, \quad \varphi^{(22)}(u^{(2)}) = \frac{\alpha_{22}u^{(2)} + \gamma_{22}(u^{(2)})^3}{\eta_{22} + (u^{(2)})^2}. \end{aligned} \tag{3.20}$$

Suppose that the following interval estimates are known a priori:

$$\begin{gathered} 18 \le \alpha_{11} \le 20, \quad 4 \le \gamma_{11} \le 6, \quad 2 \le \eta_{11} \le 3, \\ 1/5 \le \alpha_{12} \le 1/2, \quad 1/12 \le \gamma_{12} \le 1/7, \quad 1/6 \le \eta_{12} \le 1/3, \\ 1/2 \le \alpha_{21} \le 1, \quad 1/15 \le \gamma_{21} \le 1/9, \quad 1/4 \le \eta_{21} \le 1/3, \\ 30 \le \alpha_{22} \le 35, \quad 7 \le \gamma_{22} \le 9, \quad 2 \le \eta_{22} \le 5. \end{gathered} \tag{3.21}$$

According to (3.19) and (3.20), taking into account (3.21), we find

$$\begin{gathered} 3.2 < \partial\phi^{(1)}/\partial u^{(1)} \le 10, \quad -3 \le \partial\phi^{(1)}/\partial u^{(2)} < 0.32, \\ -0.44 < \partial\phi^{(2)}/\partial u^{(1)} \le 4, \quad 5.5 < \partial\phi^{(2)}/\partial u^{(2)} \le 17.5. \end{gathered} \tag{3.22}$$

It can be seen from (3.22) that all the partial derivatives of the vector function's $\phi(u)$ components are bounded; in this case, $\partial\varphi^{(1)}/\partial u^{(1)}$ and $\partial\varphi^{(2)}/\partial u^{(2)}$ preserve their signs for all $u = [u^{(1)}, u^{(2)}]^{\mathrm{T}}$ from R^2, while $\partial\varphi^{(1)}/\partial u^{(2)}$ and $\partial\varphi^{(2)}/\partial u^{(1)}$ reverse their signs (see Remark 2).

To illustrate this, Fig. 3 shows four nonlinearities $\varphi^{(11)}(u^{(1)})$, $\varphi^{(12)}(u^{(2)})$, $\varphi^{(21)}(u^{(1)})$, and $\varphi^{(22)}(u^{(2)})$, which belong to family (3.21), and four derivatives $\mathrm{d}\varphi^{(11)}/\mathrm{d}u^{(1)} \equiv \partial\varphi^{(1)}/\partial u^{(1)}$, $\mathrm{d}\varphi^{(12)}/\mathrm{d}u^{(2)} \equiv \partial\varphi^{(1)}/\partial u^{(2)}$, $\mathrm{d}\varphi^{(21)}/\mathrm{d}u^{(1)} \equiv \partial\varphi^{(2)}/\partial u^{(1)}$, and $\mathrm{d}\varphi^{(22)}/\mathrm{d}u^{(2)} \equiv \partial\varphi^{(2)}/\partial u^{(2)}$ in the particular case where

$$\begin{gathered} \alpha_{11} = 20, \quad \gamma_{11} = 5, \quad \eta_{11} = 2, \quad \alpha_{12} = -1/3, \quad \gamma_{12} = -1/10, \quad \eta_{12} = -1/4, \\ \alpha_{21} = 1, \quad \gamma_{21} = 1/12, \quad \eta_{21} = 1/4, \quad \alpha_{22} = 30, \quad \gamma_{22} = 7, \quad \eta_{22} = 2. \end{gathered} \tag{3.23}$$

Using estimates (3.22), as a matrix for the linear model of the plant with nonlinearities (3.20), according to (3.1), we select

$$B_0 = \begin{pmatrix} 6.6 & -1.34 \\ 1.78 & 11.5 \end{pmatrix}. \tag{3.24}$$

In this case, due to (3.5), we have

$$\begin{gathered} -3.4 \le \delta^{(11)}(u) \le 3.4, \quad -1.66 \le \delta^{(12)}(u) \le 1.66, \\ -2.22 \le \delta^{(21)}(u) \le 2.22, \quad -6 \le \delta^{(22)}(u) \le 6, \end{gathered} \tag{3.25}$$

and $\det B_0 = 78.2852 \ne 0$; thus,

$$B_0^+ = B_0^{-1} = (2500/195713)\begin{pmatrix} 11.5 & 1.34 \\ -1.78 & 6.60 \end{pmatrix}. \tag{3.26}$$

By solving the four pairs of the linear programming problems (3.13) under conditions (3.25) taking into account (3.26), we obtain

$$\min \sigma^{(11)} = -105187/195713, \quad \max \sigma^{(11)} = 105187/195713,$$

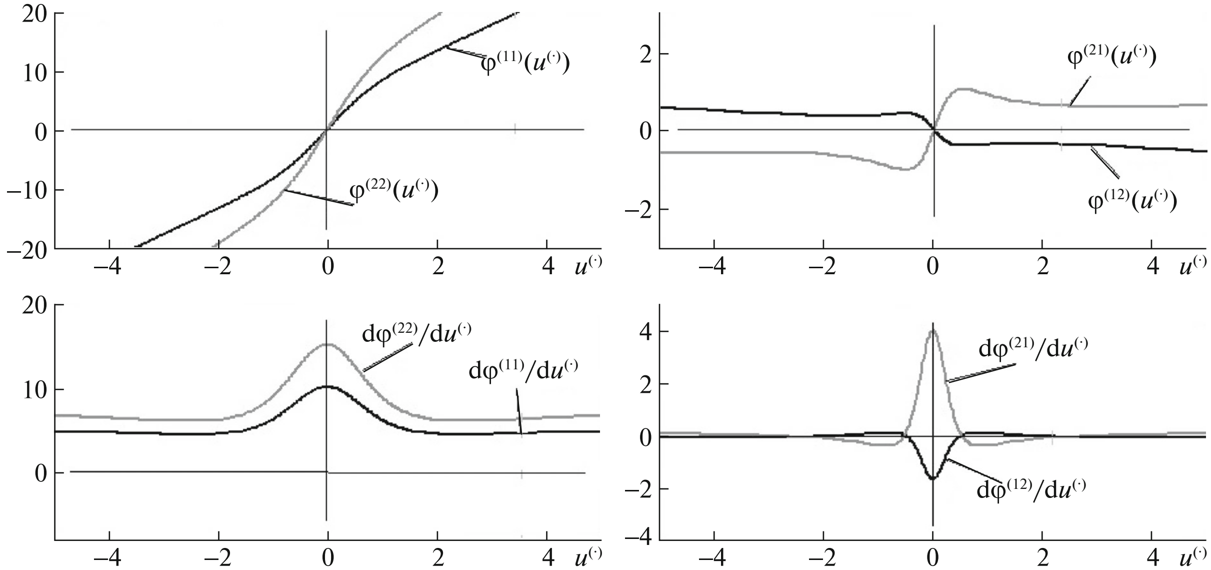

Fig. 3. Functions $\varphi^{(ij)}(u^{(\cdot)})$ of form (3.20) and their derivatives with respect to $u^{(\cdot)}$ for $u^{(\cdot)} = u^{(1)}, u^{(2)}$ in case of (3.23).

$$\min\sigma^{(12)} = -67825/195713, \quad \max\sigma^{(12)} = 67825/195713,$$

$$\min\sigma^{(21)} = -51760/195713, \quad \max\sigma^{(21)} = 51760/195713,$$

$$\min\sigma^{(22)} = -106387/195713, \quad \max\sigma^{(22)} = 106387/195713.$$

Hence,

$$\sum_{i=1}^{2}\max\{|\min\sigma^{(1i)}|, |\max\sigma^{(1i)}|\} = 173012/195713 \approx 0.884,$$
$$\sum_{i=1}^{2}\max\{|\min\sigma^{(2i)}|, |\max\sigma^{(2i)}|\} = 158147/28903 \approx 0.808. \quad (3.27)$$

Expressions (3.27) directly imply that the conditions in the corollary of Theorem 3 are fulfilled. This brings us to the following conclusion: if the matrix B_0 has form (3.24), then a closed-loop control system comprising plant (1.1) with the nonlinearity described by (3.19), (3.20) and a linear controller (1.9), (2.11) with matrix B_0^+ form(3.26) is robustly stable for the whole family of nonlinearities given by the interval estimates (3.21).

To support this conclusion, the control system (1.1), (1.9), and (2.11) with nonlinearity parameters (3.23) was simulated under $v_n^{(1)}, v_n^{(2)} \equiv 0$ and $v_n^{(1)}, v_n^{(2)} \not\equiv 0$. In the simulation experiments, we assumed that $y^0 = [5, 13]^{\mathrm{T}}$ and $u_0 = [0, 0]^{\mathrm{T}}$. The disturbance sequences $\{v_n^{(1)}\}$, $\{v_n^{(2)}\}$ were simulated as sequences of pseudorandom numbers uniformly distributed on the interval $[-1, 1]$.

The results of the experiments (each being 50 clock periods long) under zero and limited disturbances are presented in Figs. 4 and 5, respectively, where $\|u_n\|_2 = \sqrt{(u_n^{(1)})^2 + (u_n^{(1)})^2}, \|y_n\|_2 = \sqrt{(y_n^{(1)})^2 + (y_n^{(1)})^2}$, $\|y^0\|_2 = \sqrt{(y^{0(1)})^2 + (y^{0(1)})^2}$.

Figure 4 shows that the linear pseudoinverse-based controller (1.9) and (2.11) ensures the stability of the control system over the nonlinear plant (1.1) under interval uncertainties about the parameters of nonlinearity $\varphi(u)$. Figure 5 shows that the controller successfully copes with uncontrollable bounded disturbances, keeping the vector of output variables y_n in the neighborhood of the point y^0.

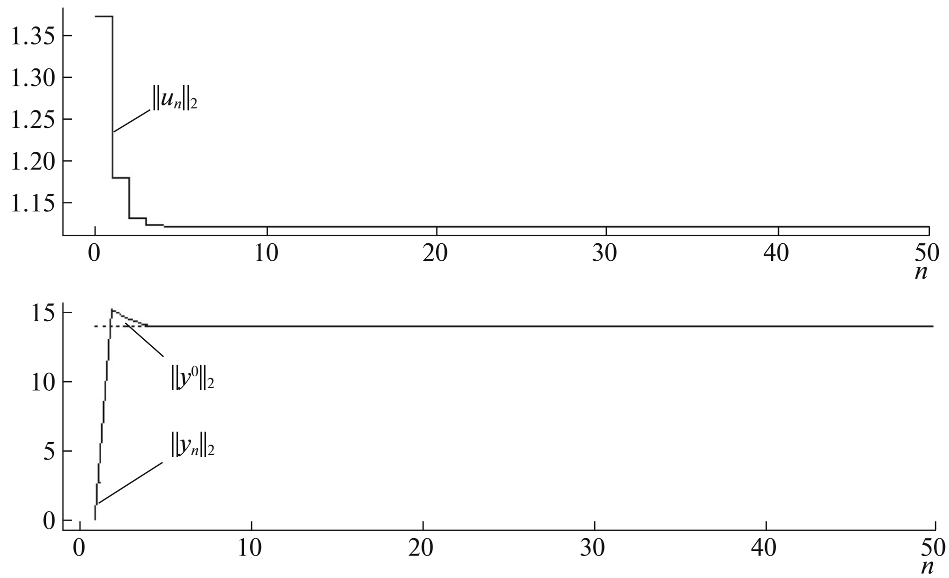

Fig. 4. Behavior of control system under zero disturbances.

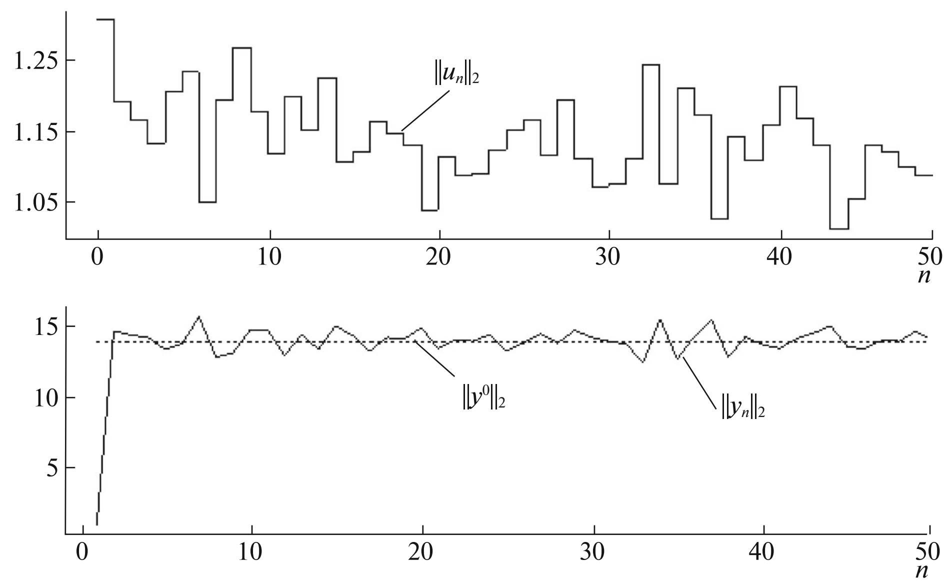

Fig. 5. Behavior of control system under limited disturbances.

CONCLUSIONS

In this paper, a robust stabilization problem has been considered for a particular class of nonlinear multivariable discrete-time time-invariant plants subjected to nonmeasurable bounded additive disturbances. It is assumed that the steady-state characteristics of these plants are a priori uncertain, and only the boundaries of the possible values of the elements in the corresponding Jacobian matrices are known, with these boundaries having the same sign for all elements of these matrices. A controller has been designed that consists of two series-connected blocks: a linear pseudoinverse model with fixed parameters and a discrete-time integrator. Sufficient conditions for the robust stability of a closed-loop linear-feedback control system over uncertain nonlinear time-invariant plants have been formulated. The estimates have been obtained for the dissipativity region of these systems under arbitrary bounded disturbances.

A numerical example and simulation results have been presented that illustrate the proposed approach to designing robust control systems based on fixed linear models of nonlinear multivariable time-invariant plants operating under conditions of limited uncertainty.

APPENDIX

Proof of Theorem 3. Based on (1.1), (1.9), and (2.11), due to (2.17), the motion equation of the system in deviations from the equilibrium u^e takes the form

$$u_{n+1} - u^e = u_n - u^e - B_0^+(\varphi(u_n) - \varphi(u^e)) - B_0^+ v_n. \tag{A.1}$$

By the condition of the theorem, $\varphi(u)$ is a function differentiable with respect to the components $u^{(1)}, \ldots, u^{(r)}$ of the vector u. Therefore, according to [19, p. 17], the following relation for its increment holds:

$$\varphi(u_n) - \varphi(u^e) = \int_0^1 B(u^e + \theta(u_n - u^e))(u_n - u^e)\mathrm{d}\theta, \tag{A.2}$$

where $B(u)$ is a matrix of form (2.1) with the elements $b^{(ij)}(u) = \partial\varphi^{(i)}(u)/\partial u^{(j)}$, $i = \overline{1, m}$, $j = \overline{1, r}$.

By substituting (A.2) into (A.1), we obtain

$$u_{n+1} - u^e = \int_0^1 (I - B_0^+ B(u^e + \theta(u_n - u^e)))(u_n - u^e)\mathrm{d}\theta - B_0^+ v_n. \tag{A.3}$$

With (A.3) and $\|I - B_0^+ B_0\| = 0$ (due to Lemma), we can write

$$\begin{aligned} \|u_{n+1} - u^e\|_\infty &\le \max_{\Delta(u):\, \delta^{(ij)}(u) \in [\underline{\delta}^{(ij)}, \overline{\delta}^{(ij)}]} \|B_0^+ \Delta(u)\|_\infty \|u_n - u^e\|_\infty \\ &+ \|B_0^+\|_1 \sup_{0 \le n < \infty} \|v_n\|_\infty \le q\|u_n - u^e\|_\infty + \|B_0^+\|_1 \varepsilon. \end{aligned} \tag{A.4}$$

Here, q is given by expression (3.12), while

$$\varepsilon := \sup_{0 \le n < \infty} \|v_n\|_\infty = \max_{i = \overline{1, r}} \varepsilon^{(i)} \tag{A.5}$$

(due to constraints (1.6) and by the definition of the ∞ norm [18, Subsect. 14.26]; see also [17, p. 260]).

Now, with (A.4) and $\varepsilon < \infty$, we obtain the chain of inequalities

$$\begin{aligned} \|u_{n+1} - u^e\|_\infty &\le q\|u_n - u^e\|_\infty + \|B_0^+\|_1 \varepsilon \le \ldots \le q^n \|u_0 - u^e\|_\infty \\ &+ (q^{n-1} + \ldots + q + 1)\|B_0^+\|_1 \varepsilon < \infty. \end{aligned} \tag{A.6}$$

Based on (A.6), we can conclude that if condition (3.11) is fulfilled, then the ∞ norm of deviations of u_n from u^e for $n \to \infty$ is, in the limiting case, an upper-bounded sum of a certain infinitely decreasing geometric sequence with a denominator q that is defined by the right-hand side of (3.17). To complete the proof, we only need to make sure that relation (3.18) holds. For this purpose, Eq. (1.1) is rewritten as

$$y_n - y^e = \varphi(u_n) - \varphi(u^e) + v_n, \tag{A.7}$$

taking into account that $y^e = \varphi(u^e)$. Then, using the inequality

$$\|\varphi(u_n) - \varphi(u^e)\|_\infty \le \max_{B:\, \underline{b}^{(ij)} \le b^{(ij)} \le \overline{b}^{(ij)}} \|B\|_1 \|u_n - u^e\|_\infty,$$

which is implied by (A.2), based on (A.7), and taking into account (A.5) and the well-known triangle inequality [18, Subsect. 14.20], we obtain

$$\|y_n - y^e\|_\infty \le \max_{B:\, \underline{b}^{(ij)} \le b^{(ij)} \le \overline{b}^{(ij)}} \|B\|_1 \|u_n - u^e\|_\infty + \varepsilon.$$

Due to (3.17), this leads directly to relation (3.18). The theorem is proved.

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